Evolution equations of a stabilized helical pinch

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Equations are derived to describe the reversed profile of the longitudinal magnetic field averaged over a magnetic surface, in a helical pinch.

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In a "stabilized" pinch with a longitudinal magnetic field which reverses at the plasma boundary, the safety factor (or rotation number) q has the monotonic profile required for plasma stability.[1] The reversal of the field and thus of q is automatically maintained by the turbulent excitation of azimuthal currents outside the pinch.

Ohkawa *et al.*[2] have recently shown that q reversal can be achieved even without external azimuthal currents by deforming the surface of the pinch into a helix. The result is a magnetic configuration much like that of a stellarator field, except that the longitudinal field in the pinch is weak, and the ohmic-heating current is relatively high ($q \ll 1$). We should emphasize that the point of this suggestion of Ohkawa *et al.* is not a local reversal of the longitudinal field at each point along the azimuth but a subtler effect, comparable to the rotational transform in a stellarator.

If a stable helical-pinch configuration is to be maintained for a long time, it is important to determine whether the q reversal is compatible with a laminar Ohm's law. Ohkawa *et al.*[2] did not take up this question. We know that in a cylindrical pinch the reversal of the field (and q) is incompatible with the ordinary Ohm's law,[1] and turbulent mechanisms for the generation of the poloidal current must be taken into account, as has been mentioned previously.[3]

To determine whether the q reversal is compatible with the ordinary Ohm's law, we consider the equilibrium equations

$$\nabla p = [\mathbf{jB}], \quad \mathrm{rot}\,\mathbf{B} = \mathbf{j}, \quad \mathrm{div}\,\mathbf{B} = 0, \tag{1}$$

supplemented with Maxwell's equation for the electric field,

$$\mathrm{rot}\,\mathbf{E} = -\partial \mathbf{B}/\partial t, \tag{2}$$

and with the projection of the generalized Ohm's law on the magnetic field,

$$\mathbf{j}\,\mathbf{B} = \sigma_{\parallel}\mathbf{EB}. \tag{3}$$

These equations are adequate for describing the plasma, according to Refs. 4 and 5, if the calculations for the transport across the magnetic field are replaced by the assumption of a given distribution of the plasma pressure p and of the longitudinal conductivity $\sigma_{\parallel}$.

In order to describe a two-dimensional configuration with helical symmetry, we must work with the integral characteristics of the currents and fluxes. We denote by Ψ and F the flux and current which are external with respect to the magnetic surface

$a(r,\theta) = \text{const}$ and which cross the helicoidal surface $\theta = \omega - az(r, \omega, z$ are cylindrical coordinates). In a straight helical pinch, this flux and current are related to the longitudinal current j and to the flux Φ inside the magnetic surface $a = \text{const}$ by

$$j = -a_{22}\Psi' + a_{23}\Phi'; \; F = -a_{23}\Psi' + a_{33}\Phi', \tag{4}$$

where the prime denotes differentiation with respect to a, and the α_{ik} are coefficients determined by the cross-sectional shape of the magnetic surfaces,[6]

$$a_{33} = L^2\left[\frac{dV}{da}\left\langle\frac{1}{1+a^2r^2}\right\rangle\right]^{-1}, \quad a_{23} = \frac{2a}{L^2}a_{33}\int\frac{dV}{(1+a^2r^2)^2},$$

$$a_{22} = \frac{1}{L^2}\frac{dV}{da}\left\langle\frac{|\nabla a|^2}{1+a^2r^2}\right\rangle + \frac{a_{23}^2}{a_{33}}. \tag{5}$$

Here V is the volume enclosed by the magnetic surface, L is the total length of the system, and the angle brackets denote the average over the layer between closely spaced magnetic surfaces. In addition to the functions Ψ and F, which are conventionally used to describe helically symmetric systems, we introduce the external poloidal flux Ψ_ω and the external poloidal current I across a straight membrane $\omega = \text{const}$,

$$\Psi_\omega = \Psi - N\Phi, \; I = F - Nj \tag{6}$$

in order to take explicit account of the "twisting" of the system. Here $N = \alpha R$ ($2\pi R = L$), and $|N|$ is the number of periods in the system. From (4) and (6) we find

$$\Phi' = \Lambda(I + \mu j) \tag{7}$$

where $\Lambda = a_{22}(a_{22}a_{23} - a_{23}^2)^{-1}$, and $\mu = N - \dfrac{a^{23}}{a^{22}}$. We note that the quantity μ is the rotational transform of a stellarator without a current, with magnetic surfaces of the same shape as those of the pinch under consideration. It can be seen from (7) that if $\mu \neq 0$, i.e., for configurations with magnetic surfaces of noncircular cross section, the longitudinal flux Φ depends explicitly on the longitudinal current j. This dependence is a manifestation of the "translational transform,"[2] which—like the analogous rotational transform in a stellarator—is introduced by the helical external fields. It follows from (7) that this effect makes it possible to reverse $q = -\Phi'/\Psi'_\omega$ without reversing the homogeneous longitudinal external field, i.e., without reversing I, because of the term μj.

Using the equation

$$\Phi'\dot{\Psi} - \Psi'\dot{\Phi} = \frac{1}{\sigma_\parallel}(jF' - Fj'), \tag{8}$$

which is an integral consequence of (3) (the dot denotes differentiation with respect to the time), we find evolution equations for the longitudinal current j and for the average field over a magnetic surface, $\bar{B}_z = \Phi'/2\pi a$, in a pinch with an external voltage ϵ_0. If the poloidal flux in the plasma is assumed to remain constant, then $\epsilon_0 = -\dot{\Psi}_\omega$, and

$$j' = -\frac{p'j}{\langle B^2\rangle} + \sigma_\parallel\frac{\Phi'^2(\epsilon_0 - \dot{\Phi}/q)}{\langle B^2\rangle V'}, \tag{9}$$

$$\bar{B}_z' = -\left[\frac{p'}{<B^2>} + \frac{\Lambda}{a}\left(\frac{a}{\Lambda}\right) + \sigma_{\parallel}\frac{j\,(\epsilon_0 - \dot{\Phi}/q\,)\,\Lambda}{<B^2>V'}\,\frac{}{a_{22}}\right]\bar{B}_z + \Lambda\mu'\,\frac{j}{2\pi a}\,\cdot \tag{10}$$

The last term in (10), which explicitly takes into account the twisting of the system and the noncircular shape of the $z =$ const cross sections of the magnetic surfaces, serves as the generator of the field $\bar{B}_z$. Simplified models will not suffice for calculating this term. In an $l = 2$ configuration, for example, it is necessary to consider such a weak effect as the dependence of the eccentricity of the cross sections of the magnetic surfaces on their average radius. If the cross sections are only slightly noncircular, this term is small, but in the limit $\bar{B}_z \rightarrow 0$ it becomes the leading term, and only this term can lead to the reversal of $\bar{B}_z$. At $\bar{B}_z = 0$, the derivative $\bar{B}_z'$ has a finite value (if $\mu' \neq 0$).

It thus follows from expression (10) for the effective longitudinal-field generation that q reversal can occur in a helical pinch and that this reversal is compatible with the ordinary Ohm's law. This effect can occur only if the "rotational transform" μ is inhomogeneous; this transform is determined by the cross-sectional shape of the magnetic surfaces. Reversal of q does not require reversal of the longitudinal field.

Equations (9) and (10) can also be used to describe the transient events leading to the reversed profile in a helical pinch.

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