Use of a regularization of infinite determinants to study parametric instabilities

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A new method is proposed for studying parametric instabilities in a homogeneous magnetized plasma. This method makes it possible to reduce the problem of finding all the natural modes to one of finding the roots of a polynomial, without invoking small parameters. The instability growth rate is derived explicitly for the two-wave interaction. The use of this method is demonstrated through a determination of the growth rate for the modulational instability of a lower hybrid wave.

In general, the problem of finding the growth rates of a parametric instability of a wave $\mathbf{E}(t)=\mathbf{E}_0\sin(\Omega t)$ in a homogeneous plasma in a static magnetic field reduces to the problem of finding values of ω at which the following infinite determinant vanishes:[1]

$$D_N=\left|\begin{array}{c|c} I_{mn} & D^e_{mn} \\ \hline D^i_{mn} & I_{mn}\end{array}\right|. \tag{1}$$

Here I is the unit matrix ($I_{mn}=\delta_{mn}$, where

$$\delta_{mn}=\begin{cases}1, & m=n,\\ 0, & m\neq n\end{cases}$$

is the Kronecker delta), $D^e_{mn}=R_{em}J_{n-m}(\mu)$, $D^i_{mn}=R_{im}J_{m-n}(\mu)$, $\mu=|\mathbf{k}\cdot\mathbf{r}|$, $\mathbf{k}$ is the wave vector of an unstable wave, $\mathbf{r}$ is the amplitude of the electron oscillations in the pump field,

$$R_{\alpha m}=\frac{\delta\epsilon_\alpha(\omega+m\Omega,\mathbf{k})}{1+\delta\epsilon_\alpha(\omega+m\Omega,\mathbf{k})},$$

$\delta\epsilon_\alpha(\omega+m\Omega,\mathbf{k})$ is the contribution of the particles of species α to the longitudinal permittivity of the plasma, and J_n is the Bessel function of the first kind of index n.

This problem can be solved analytically if the arguments of the Bessel functions, μ, are small, and it is sufficient to deal with the harmonics $m=0, \pm 1$. Otherwise, it becomes necessary to study the determinant of a large-rank matrix; that problem cannot be handled analytically. A numerical determination of the eigenvalues of this determinant, whose elements depend on ω, is a complex problem. An effective way to study infinite determinants which arise in a study of second-order differential equations with periodic coefficients was proposed by Hill.[2] If the matrix elements have no singularities other than simple poles in the complex ω plane, the infinite determinant, being an analytic, meromorphic, periodic function of ω, can be expressed as the sum of (a) a simple periodic function which has poles at the same points, and with the same principal parts, as the determinant, and (b) an entire function. If we are interested in electrostatic waves in a cold, magnetized plasma, then the determinant of system (1), $D(\omega)$, while differing from that analyzed by Hill, does have some similar features: $D(\omega)$ is an even periodic function of ω with a period Ω, and its poles, which are the roots of the equation $1+\delta\epsilon_\alpha(\omega+n\Omega,\mathbf{k})=0$, constitute eight sequences $\omega^*=\pm\omega_{\alpha m}+n\Omega$, $n=0, \pm1, \pm2,\ldots$, where

$$\omega_{\alpha m}^2=\frac{\omega_{p\alpha}^2+\Omega_{H\alpha}^2}{2}+\frac{(-1)^m}{2}[(\omega_{p\alpha}^2+\Omega_{H\alpha}^2)^2-4\Omega_{H\alpha}^2\omega_{p\alpha}^2\cos^2\theta]^{1/2}. \tag{2}$$

Here θ is the angle between the wave vector $\mathbf{k}$ and the magnetic field, $\omega_{p\alpha}$ is the Langmuir frequency of the particles of species α, $\Omega_{H\alpha}$ is the cyclotron frequency of these particles, and m takes on the values 0, 1.

We write $D(\omega)$ in the form

$$D(\omega)=N(\omega)+\sum_\alpha\sum_m\frac{K_{\alpha m}}{\sin^2(\pi\omega_{\alpha m}/\Omega)-\sin^2(\pi\omega/\Omega)}, \tag{3}$$

where

$$K_{\alpha m}=-\frac{\pi\sin(2\pi\omega_{\alpha m}/\Omega)}{\Omega}\lim_{\delta\to0}\delta D(\omega_{\alpha m}+\delta)=\frac{\pi D_{\alpha m}\sin(2\pi\omega_{\alpha m}/\Omega)}{\Omega\partial\delta\epsilon_\alpha(\omega,\mathbf{k})/\partial\omega|_{\omega=\omega_{\alpha m}}}, \tag{4}$$

and $D_{\alpha m}$ is the determinant of the matrix $D(\omega=\omega_{\alpha m})$, in which the row containing the singularity at $\omega=\omega_{\alpha m}$ has been regularized: The 1 in this row has been replaced by 0, and the $R_{\alpha m}$ by 1. With $K_{\alpha m}$ chosen in this manner, the double sum on the right side of (3) has principal parts which coincide with $D(\omega)$ at all the poles of $D(\omega)$, so $N(\omega)$ has no poles anywhere in the complex ω plane. In other words, it is an entire periodic function of ω. It is also easy to see that $N(\omega)$ is bounded, since we have $N(\omega=i\infty)=D(\omega=i\infty)=1$, and according to the Liouville theorem it is identically equal to a constant[3] $[N(\omega)\equiv1]$. The dispersion relation for this case is

$$D(\omega)=1+\sum_\alpha\sum_m\frac{K_{\alpha m}}{\sin^2(\pi\omega_{\alpha m}/\Omega)-\sin^2(\pi\omega/\Omega)}=0, \tag{5}$$

where $K_{\alpha m}$ is given by (4). This equation reduces to a polynomial in $\sin(\pi\omega/\Omega)$. We wish to stress that Eq. (5) is exact, and the errors in the calculated eigenfrequencies are determined by the errors in the calculations of $K_{\alpha m}$, since there are no difficulties in numerically finding all the roots of a polynomial of any degree. The problem can be simplified substantially by considering the interaction between two wave branches, in which we need to consider one root in each of the equations $1+\delta\epsilon_\alpha(\omega+n\Omega,\mathbf{k})=0$, e.g., the case in which the magnetic field is zero or infinite. In this case the dispersion relation is biquadratic in $\sin(\pi\omega/\Omega)$ and can easily be solved for ω:

$$\omega=\pm\frac{\Omega}{\pi}\arcsin\left[\left(\frac{A\pm(A^2-4B)^{1/2}}{2}\right)^{1/2}\right], \tag{6}$$

where

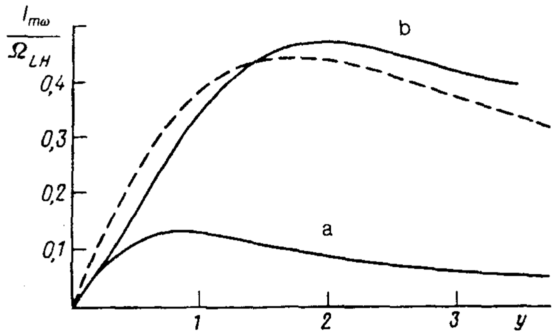

FIG. 1. Instability growth rate versus y. a—$\mu=0.3$; b—$\mu=2$; dashed curve—results calculated from Eq. (7).

$$A=K_e+K_i+\sin^2(\pi\omega_e/\Omega_{\rm LH})+\sin^2(\pi\omega_i/\Omega_{\rm LH}),$$

$$B=K_e\sin^2(\pi\omega_i/\Omega_{\rm LH})+K_i\sin^2(\pi\omega_e/\Omega_{\rm LH})+\sin^2(\pi\omega_e/\Omega_{\rm LH})\sin^2(\pi\omega_i/\Omega_{\rm LH}).$$

If small parameters are available, an analytic solution can be found with any degree of accuracy through an expansion of A and B in these parameters. Let us consider the application of this method to the problem of the modulational instability of a wave at the lower hybrid frequency[4–8] $\Omega=\omega_{pi}/(1+\omega_{pe}^2/\Omega_{He}^2)^{1/2}$. An application of the method to this problem is interesting because the maximum of the instability growth rate approaches the pump frequency at large amplitudes of the pump wave, and it reaches the pump frequency at[7] $\mu\gg 1$. In this case the system has no small parameters, and a large number of harmonics must be taken into account in order to solve the problem correctly.

Let us consider the instability of a wave whose wave vector **k** is nearly perpendicular to the direction (z) of the magnetic field: $k_z/k\ll\omega_{pi}/\omega_{pe}$. Under the conditions

$$\Omega_{Hi}\ll\omega\ll\Omega_{He},\quad \omega_{pe}\ll\Omega_{He},\quad \omega/k_z v_{Te}\gg 1,\quad kr_{de(di)}\ll 1,$$

where $r_{de(di)}$ is the electron (ion) Debye length, and the increments in the permittivity of the plasma for waves with $k_z/k\simeq\omega_{pe}/\omega_{pi}$ are $\delta\epsilon_e=-\omega_{pi}^2y^2/\omega^2$ and $\delta\epsilon_i=-\omega_{pi}^2/\omega^2$, where $y=k_z\omega_{pe}/k\omega_{pi}$. The poles of the determinant $D(\omega)$ are $\omega_{en}=\pm\Omega_{\rm LH}y+n\omega_{pi}$ and $\omega_{in}=\pm n\omega_{pi}$. A distinctive feature of this system is that the frequency of the pump wave, $\Omega=\omega_{pi}$, coincides with a root of the equation $1+\delta\epsilon_i(\omega,\mathbf{k})=0$, and the determinant has second-order poles at $\omega=\omega_{in}$. In this case, solution (6) remains in force, except that we have

$$K_i=-\pi^2\lim_{\delta\to 0}[\delta^2D(\omega_{am}+\delta)]/\Omega^2.$$

Figure 1 shows the growth rate calculated from (6) as a function of y for various values of μ. The solid curve is the y dependence of the instability growth rate, while the dashed curve is the result of a solution of the equation

$$\frac{1}{1+\delta\epsilon_e(\omega,\mathbf{k})}+\frac{1}{\delta\epsilon_i(\omega,\mathbf{k})}+\frac{\mu^2}{4}\left(\frac{1}{1+\delta\epsilon_e(\omega+\Omega,\mathbf{k})+\delta\epsilon_i(\omega+\Omega,\mathbf{k})}\right.$$
$$\left.+\frac{1}{1+\delta\epsilon_e(\omega-\Omega,\mathbf{k})+\delta\epsilon_i(\omega-\Omega,\mathbf{k})}\right)=0.$$

This equation is ordinarily used to analyze this instability.[4–8] Its range of applicability is limited to values $\mu \ll 1$. For the small parameter value $\mu=0.3$ (curve a in Fig. 1) the results of the numerical calculations are essentially the same as the results calculated from Eq. (7). At $\mu=2$ (curve b in Fig. 1), Eq. (7) generates incorrect results (which are shown by the dashed curve in this figure). The number of harmonics which must be taken into account in order to reach the required accuracy increases with increasing μ ($N=6$ at $\mu=0.3$ and $N=15$ at $\mu=2$).

The method proposed here makes it a simple matter to find the growth rates for parametric plasma instabilities without the use of small parameters. This method can also be used to solve problems in which infinite periodic determinants arise. Examples are problems concerning the propagation of waves and electron beams in periodic structures and the scattering of electromagnetic waves by relativistic electron beams.

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