Random resistor network as a (−1)-component Landau–Ginzburg model

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The critical behavior of a resistor network, from which elements are cut out at random, is described by critical exponents of an O_n-symmetry, n-component φ^4 Euclidean field theory with $n=-1$.

The critical behavior of resistor networks from which elements are cut out at random has continued to hold interest for several years.[1,12] On the basis of an analogy with percolation problems, it has been hypothesized[3] that the upper critical dimensionality in this problem is $d_c=6$. This hypothesis later found support in the results of high-temperature expansions.[4] The critical exponents at $d>6$ are therefore canonical ($\eta=0$, $\nu=1/2$), while those at $d<6$ are calculated by the renormalization-group method in the form of $(6-\epsilon)$ expansions in the parameter $\epsilon=6-d$, which is the deviation of the dimensionality of the space, d, from the critical value.[5,8,10–12] So far, however, the problem cannot be regarded as completely solved. For example, certain deficiencies of the earlier studies were found in Ref. 7, while some substantial changes were made in the results of the $(6-\epsilon)$ expansions in Ref. 8.

In the present letter we show that the critical behavior of a random resistor network is described by the critical exponents of an O_n-symmetry, n-component φ^4 Euclidean field theory with $n=-1$. This assertion means, in particular, that the upper critical dimensionality of the problem is $d_c=4$, and it is a simple matter to derive a $(4-\epsilon)$ expansion of the critical exponents at $d<4$.

As we know,[2] the problem of random resistor networks can be treated as a q-site Potts model in the limit $q\to 0$ (the percolation problem corresponds to another formal special case, $q=1$). The critical behavior of the Potts model in the continuum limit is described[13] by the Euclidean theory of an $n\equiv(q-1)$-component field with an action functional

$$S(\varphi)=(\vec{\nabla}\varphi_i\vec{\nabla}\varphi_i)/2+\tau\varphi_i\varphi_i/2+ut_{ijk}\varphi_i\varphi_j\varphi_k/6 + wt_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l/24+v(\varphi_i\varphi_i)^2/24. \quad (1)$$

An integration over the d-dimensional space $\mathbf{x}$ is to be understood in (1), as is a summation from 1 to n over a repeated Latin-letter subscript. The quantity $\tau\propto T-T_c$ represents the deviation of the temperature (or analog thereof) from the critical value, while u, v, and w are parameters of the model (coupling constants). The tensor structures in (1) are given by

$$t_{ijk}=\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha, \quad t_{ijkl}=\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha, \quad (2)$$

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where $\mathbf{e}^\alpha=\{e_i^\alpha\}$ is a set of n-dimensional unit vectors (there are q such unit vectors) which satisfy the relations

$$\sum_{\alpha=1}^{n+1} e_i^\alpha=0, \quad \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha=q\delta_{ij}/n, \quad \sum_{i=1}^{n} e_i^\alpha e_i^\beta=(q\delta^{\alpha\beta}-1)/n. \tag{3}$$

The action in (1) corresponds to a standard quantum-field diagram perturbation theory with a propagator $\delta_{ik}(k^2+\tau)^{-1}$, with the triple vertex ut_{ijk}, and with the two quadrupole vertices wt_{ijkl} and vI_{ijkl}, where $I_{ijkl}=(1/3)(\delta_{ij}\delta_{kl}+$ permutations). We will call these the "u, w, and v vertices," respectively. With $u=w=0$, model (1) becomes the O_n-symmetry Landau–Ginzburg model. Accordingly, only the v vertices contribute to any Green's function of model (1), and this function is the same as the corresponding Green's function of the Landau–Ginzburg model.

According to symmetry considerations, the 1-irreducible, N-point Green's functions $\Gamma^{(N)}$ of model (1) are

$$\Gamma_{ij}^{(2)}=D_{ij}^{-1}=\delta_{ij}F^{(2)}(u,w,v), \quad \Gamma_{ijk}^{(3)}=t_{ijk}F^{(3)}(u,w,v), \tag{4}$$

$$\Gamma_{ijkl}^{(4)}=I_{ijkl}F^{(4)}(u,w,v)+t_{ijkl}F^{(5)}(u,w,v).$$

The dependence of the scalar functions $F^{(N)}$ on u, w, and v is indicated explicitly, while a dependence on τ and on variables of the coordinate or momentum type is to be understood. Model (1) was originally defined for natural values of n. The n dependence enters the diagrams through symmetry coefficients, which, after the tensor factors [of the type shown explicitly in (4)] are singled out, are always polynomials in n and can be generalized in an obvious way to arbitrary n. In this sense we can speak in terms of a Landau–Ginzburg model with $n=0$ (the problem of non-self-intersecting polymer chains), model (1) with $n=0$ or $n=-1$, etc.

The critical exponents of model (1) and of the related ATR model were calculated in Ref. 14 in the form of $(6-\epsilon)$ and $(4-\epsilon)$ expansions. The latter case corresponds to a "tricritical" behavior with $u=0$ [in the case of the $q=2$ Ising model, the cubic contribution to (1) disappears identically]. Although the critical exponent η calculated in the $(6-\epsilon)$ expansion has a finite limit as $q\to 0$, the corresponding fixed point of the renormalization group diverges: $u_* \propto q^{-2}$. In other words, the critical regime itself disappears. Accordingly, the $q=0$ case must be analyzed separately. In this case, relations (3) become

$$\sum_{\alpha=1}^{n+1} e_i^\alpha=0, \quad \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha=0, \quad \sum_{i=1}^{n} e_i^\alpha e_i^\beta=1. \tag{5}$$

It follows from (5) that nearly all contractions of tensors (2) disappear in the case $n=-1$. For example, we have

$$t_{iij}=0, \; t_{iijk}=0, \quad t_{ijk}t_{ilm}=0, \quad t_{ijk}t_{ilms}=0. \tag{6}$$

The sole exceptional case is the six-sign entity $t_{ijkl}t_{imsp}=t_{jkl}t_{msp}$. In the construction of diagrams, the vector indices of tensors (2) contract with each other and with the δ-symbols of the propagators and the v vertices. Any diagram $\Gamma^{(2)}$ which contains at least one u or w vertex disappears, since it is not possible to find a tensor δ_{ik} in a convolution involving at least one of tensors (2), because of (6). Consequently, the function $F^{(2)}$ in (4) depends on v alone, and it is the same (see the discussion above) as the corresponding function of the O_n-symmetry Landau–Ginzburg model with $n=-1$ (we recall that we have $n=q-1$ and that for a resistor network we would have $q=0$). Using (6), we thus see that any diagram $\Gamma^{(3)}$ is zero, unless it has precisely one u vertex and no w vertices, i.e., $F^{(3)}=uf^{(3)}(v)$. The only nonzero contribution to $\Gamma^{(4)}$ comes from diagrams which lack u and w vertices (a contribution of the I_{ijkl} type) and diagrams with a single w vertex and no u vertices (a contribution of the t_{ijkl} type); i.e., we have $F^{(4)}=F^{(4)}(v)$ and $F^{(5)}=wf^{(5)}(v)$. Nonzero diagrams with two w vertices arise only in $\Gamma^{(6)}$ [see the comment following Eq. (6)].

We have thus shown that the propagator $D=\langle\varphi(\mathbf{x})\varphi(\mathbf{x}')\rangle$ of model (1) with $q=0$ is the same as the propagator of the O_n-symmetry Landau–Ginzburg model with a $v(\varphi^2)^2$ interaction for an n-component field with $n=-1$. The first term in $\Gamma^{(4)}$ contains only diagrams with v vertices, so it is the same as the corresponding Green's function of a Landau–Ginzburg model with $n=-1$.

The critical behavior of the Landau–Ginzburg model is well known (see Ref. 15, for example). The critical asymptotic behavior of the propagator in the case $v>0$ is

$$D(r)=C_1\, r^{-d+2-\eta} f(C_2 \tau r^{1/\nu}), \tag{7}$$

where $r=|\mathbf{x}-\mathbf{x}'|$, and C_i are normalization constants which depend on v. For $d>d_c=4$ the critical exponents are the same as the canonical ones ($\eta=0$, $\nu=1/2$), while for $d<4$ they are nontrivial. They are calculated as $(4-\epsilon)$ expansions. These conclusions remain in force for our own particular case, $n=-1$, since the corresponding fixed point of the renormalization group remains finite at $n=-1$. In particular, setting $n=-1$ in the known $(4-\epsilon)$ expansions of the critical exponents of the Landau–Ginzburg model, we find

$$\eta=\frac{\epsilon^2}{2\times 7^2}+\frac{215\epsilon^3}{8\times 7^4}+\frac{\epsilon^4}{32\times 7^6}\{29573-45696\zeta(3)\}+O(\epsilon^5), \tag{8}$$

$$\nu=\frac{1}{2}+\frac{\epsilon}{4\times 7}+\frac{19\epsilon^2}{4\times 7^3}+\frac{\epsilon^3}{32\times 7^5}\{4061-11424\zeta(3)\}+O(\epsilon^4),$$

where $\epsilon=4-d$, and $\zeta(3)=1.202\ 05$ is the Riemann zeta-function. For 3D space (i.e., for $\epsilon=1$) we find $\eta=0.0147$ and $\nu=0.5316$ from (8).

In this problem, propagator (7) represents the resistance of the part of the resistor network between the points $\mathbf{x}$ and $\mathbf{x}'$ (Refs. 1 and 4). The only experimental results which are available are for the critical exponent μ in the relation (network conductivity) $\propto \tau^{\mu}$. According to Ref. 9, for example, we have $\mu=1.9\pm 0.2$. Expressing μ in terms of standard critical exponents η and ν is not a trivial problem, and so far it has no generally accepted solution.[1,4,6,7] For example, the relation

$$\mu=[\nu(3d-4)-\beta]/2, \quad \beta\equiv\nu(d-2+\eta)/2,$$

proposed in Ref. 6, has been refuted by the results of numerical simulations.[7] At any rate, that is a separate problem, which must be solved outside the framework of any specific model used to calculate the exponents η and ν themselves in propagator (7).

We have accordingly presented some numerical values for these exponents, but we will not calculate μ in terms of them by invoking some specific representation of the type in Ref. 6.

The justification of a value $d_c=4$ for the critical behavior of random resistor networks, at least for a description of this behavior by the field theory in (1) with $q=0$, should thus be regarded as the basic qualitative result of this study. The difference between the value derived here for d_c and the value $d_c=6$ for the related percolation problem should not be surprising: A similar situation has already been revealed for the problem of non-self-intersecting polymer chains ($d_c=4$) and the related problem of "genuine" self-avoiding random walks[16] ($d_c=2$). In the case at hand, the reason for the difference is that the triple vertex in model (1) for the case $n=-1$ does not contribute to the binary correlator in which we are interested, (7), while in a model with $n=0$ all the vertices make nontrivial contributions to any Green's function.

[1] S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).

[2] C. M. Fortuin and P. W. Kasteleyn, J. Phys. Soc. Jpn. Suppl. **26**, 11 (1969); Physica **57**, 536 (1972).

[3] P. G. De Gennes, J. Phys. (Paris) Lett. **37**, L1 (1976).

[4] J. P. Straley, Phys. Rev. B **15**, 5733 (1977); M. J. Stephen and G. S. Grest, Phys. Rev. Lett. **38**, 567 (1977); A. B. Harris and S. Kirkpatrick, Phys. Rev. B **16**, 542 (1977); R. Fish and A. B. Harris, Phys. Rev. Lett. **38**, 796 (1977); Phys. Rev. B **18**, 416 (1978).

[5] D. J. Wallace and A. P. Young, Phys. Rev. B **17**, 2384 (1978); C. Dasgupta *et al.*, Phys. Rev. B **17**, 1375 (1978); A. B. Harris, Phys. Rev. B **28**, 2614 (1983); A. B. Harris *et al.*, Phys. Rev. Lett. **53**, 743 (1984); Phys. Rev. Lett. **54**, 1088 (1985); T. C. Lubensky and A.-M.S. Tremblay, Phys. Rev. B **34**, 3408 (1986).

[6] S. Alexander and R. Orbach, J. Phys. (Paris) Lett. **43**, L625 (1982).

[7] J. G. Zabolitsky, Phys. Rev. B **30**, 4077 (1984); H. J. Hermann *et al.*, Phys. Rev. B **30**, 4080 (1984); D. C. Hong *et al.*, Phys. Rev. **30**, 4083 (1984); R. Rammal *et al.*, Phys. Rev. B **30**, 4087 (1984); C. J. Lobb and D. J. Frank, Phys. Rev. B **30**, 4090 (1984).

[8] T. C. Lubensky and A.-M. S. Tremblay, Phys. Rev. B **37**, 7894 (1988); J. Adler *et al.*, Phys. Rev. B **34**, 3469 (1986); J. Machta *et al.*, Phys. Rev. B **33**, 4818 (1986).

[9] B. Abeles *et al.*, Phys. Rev. Lett. **35**, 247 (1973).

[10] R. Blumenfeld *et al.*, Phys. Rev. B **35**, 3524 (1987); A. B. Harris, Phys. Rev. B **35**, 5056 (1987); Y. Park *et al.*, Phys. Rev. B **35**, 5048 (1987).

[11] A. Aharony *et al.*, Phys. Rev. B **40**, 7318 (1989); A. B. Harris and A. Aharony, Phys. Rev. B **40**, 7230 (1989); A. B. Harris *et al.*, Phys. Rev. B **41**, 4610 (1990).

[12] Y. Meir *et al.*, Phys. Rev. B **34**, 3424 (1986); J. Wang *et al.*, Phys. Rev. B **45**, 7084 (1992); A. Aharony *et al.*, Phys. Rev. B **47**, 5756 (1993); J. Adler *et al.*, Phys. Rev. B **47**, 5770 (1993).

[13] G. R. Golner, Phys. Rev. B **8**, 3419 (1973).

[14] R. K. P. Zia and D. J. Wallace, J. Phys. A **8**, 1495 (1975); R. G. Priest and T. C. Lubensky, Phys. Rev. B **13**, 4159 (1976); Phys. Rev. B **14**, 5125 (1976); D. J. Amit, J. Phys. A **9**, 1441 (1976).

[15] S. Ma, *Modern Theory of Critical Phenomena* (New York, 1976).

[16] D. J. Amit *et al.*, Phys. Rev. B **27**, 1635 (1983).

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