Equilibrium and stability of ellipsoidal plasmoids

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The equilibrium and stability of some bounded plasma formations observed in the atmosphere of the sun are analyzed. The discussion is restricted to axisymmetric configurations. The gravitational field is ignored. Exact solutions for equilibria and local conditions for their stability are derived for a particular class of current distributions. It is assumed that there are a uniform external magnetic field and an external plasma of constant pressure.

1. Equilibrium of axisymmetric plasma configurations

If the gravitational field is ignored, then the pressure p is constant on magnetic surfaces $\psi=rA_\varphi=\text{const}$, the density ρ is an arbitrary function of the coordinates, and the equilibrium magnetic field is described by the Grad–Shafranov equation.[1] In the spherical coordinate system r,θ,φ this equation is

$$\frac{\partial^2\psi}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}+II'(\psi)+r^2\sin^2\theta p'(\psi)=0, \tag{1}$$

where $p(\psi)$ is the pressure, $I(\psi)=r\sin\theta B_\varphi$ is the current function, and $J_\varphi=II'/r\sin\theta+r\sin\theta p'$ is the azimuthal component of the current density.

For the class of configurations with $p(\psi)=p_0+p'\psi$, $I(\psi)=\alpha\psi$, Eq. 1 becomes a linear equation with a known right-hand side. Its solutions can be expressed in terms of elementary functions:[2,3]

$$\psi=\left[-\frac{p'r^2}{\alpha^2}+\sum a_n f_n(\alpha r)P_n'(\cos\theta)\right]\sin^2\theta, \tag{2}$$

where $f_n(z)=z^{1/2}J_{n+1/2}(z)$, $J_{n+1/2}(z)$ are Bessel functions, and $P_n'(x)$ are the derivatives of the Legendre polynomials. For a purely azimuthal current, $j=j_\varphi$ and $B_\varphi=0$, we have

$$\psi=\left[-\frac{p'r^4}{10}+\sum b_n r^{n+1}P_n'(\cos\theta)\right]\sin^2\theta. \tag{3}$$

A solution of Eq. 1 in the exterior volume, where there are no currents, but where there is a uniform magnetic field $\mathbf{B}_e=B_{0e}\mathbf{e}_z$, is

$$\psi_e=\left[\frac{B_{0e}r^2}{2}+\sum\frac{c_n}{r^n}P_n'(\cos\theta)\right]\sin^2\theta. \tag{4}$$

The functions $f_n(z)$ and $P_n'(x)$ are given by

$$f_1(z)=\frac{\sin z}{z}-\cos z,\quad f_2(z)=\frac{3}{z}f_1(z)-\sin z,$$

$$f_3(z)=\frac{15}{z^2}f_1(z)-\frac{6}{z}\sin z+\cos z,\ \dots\ ,$$

$$P_1'(x)=1,\quad P_2'(x)=3x,\quad P_3'(x)=(3/2)(5x^2-1),\ \dots\ .$$

2. Equilibrium and stability of plasma ellipsoids in the case $j=j_\varphi=r\sin\theta\, p'$

a. In the case of azimuthal currents, we restrict the discussion to the first harmonic in (3), and we require that it be continuous with the external field in (4) on the surface of the sphere $r=R$, $\psi=0$. We find the equilibrium function ψ to be

$$\psi=\frac{B_0}{2}\left(r^2-\frac{r^4}{R^2}\right)\sin^2\theta,\quad \psi_e=\frac{B_{0e}}{2}\left(r^2-\frac{R^3}{r}\right)\sin^2\theta,\quad B_{0e}=-\frac{2}{3}B_0, \tag{5}$$

where $B_0=B_z(0,\theta)$. In hydrodynamics the solution is known as a Hill vortex. If we instead use two harmonics from (3) (the first and the third), we find an interior solution for ψ, which vanishes at the surface of the ellipsoid. In the cylindrical coordinates r,φ,z, this solution is

$$\psi=\frac{p'}{8}\frac{1}{1+1/\nu^2}\Psi(r,z),\qquad \Psi(r,z)=r^2(2R_0^2-r^2-4z^2/\nu^2), \tag{6}$$

where $\nu=l_z/l_r$ is the ratio of the semiaxes of the elliptical cross sections of the magnetic surfaces near the magnetic axis, $r=R_0$. The boundary surface of Σ, on which we have $\psi(r,z)=0$, is an ellipsoid of revolution which passes through the point $r=R$, $z=\nu_\Sigma R$, with $R^2=2R_0^2$, $\nu_\Sigma=\nu/2$. The pressure distribution is described by

$$p=p_0+2B_0^2\left(1+\frac{1}{4\nu_\Sigma^2}\right)\frac{r^2}{R^2}\left[1-\frac{1}{R^2}\left(r^2+\frac{z^2}{\nu_\Sigma^2}\right)\right]. \tag{7}$$

The pressure reaches a maximum at the magnetic axis, $r=R_0$, and it is equal to the external pressure p_0 at the center of the ellipsoid. Figure 1 shows the magnetic surfaces for the case of a prolate ellipsoid, along with the pressure distribution in the $z=0$ plane. The function $\psi=rA_\varphi$ for the external magnetic field can be calculated by integrating over the currents. The uniform component B_{ze} is found from the requirement $B_z=0$ at the singular saddle point $r=0$, $z=\nu_\Sigma R$ of the boundary magnetic surface.

b. Equilibrium configuration (6) falls in the category of configurations with closed magnetic field lines. To study its stability, we use the necessary condition of Spies for a general geometry.[4] For the case at hand, of axial symmetry, with $j=j_\varphi$ and $B_\varphi=0$, this condition was first derived by Bernstein *et al.*[5] According to that condition, the stability region is determined by the inequality

$$\Lambda=-p'\left(\gamma p\frac{V''}{V}-p'\right)>0, \tag{8}$$

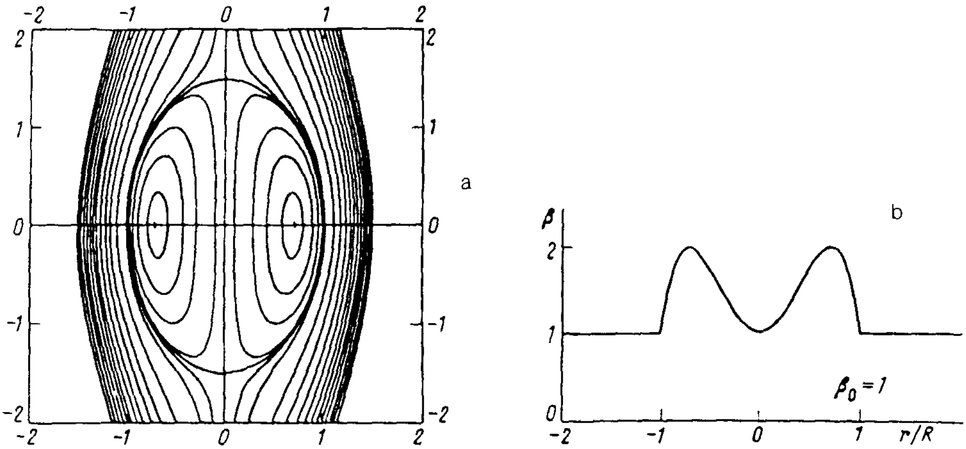

FIG. 1.

where V is the instantaneous volume, reckoned from the magnetic axis of a toroidal equilibrium configuration. The primes mean derivatives with respect to ψ, and γ is the adiabatic index. Switching to derivatives with respect to Ψ, and making the substitution $p=p_0+p'\psi$, we find

$$\Lambda=-\beta p'\left[\frac{\gamma V''(\Psi)}{V'(\Psi)}\left(1+\frac{\Psi}{\beta}\right)+\frac{1}{\beta}\right], \tag{9}$$

where

$$\beta=\frac{8p_0(1+1/\nu^2)}{p'^2R_0^4}=\frac{2p_0}{(1+1/\nu^2)B_0^2},$$

p_0 is the external pressure, and B_0 is the magnetic field at the center of the sphere. {In the cgs system we would have $\beta=8\pi p_0/[(1+1/\nu^2)B_0^2]$.} The expression for the specific volume $V'(\Psi)$ is found by integrating along the magnetic field lines:

$$V'(\Psi)=\int\frac{rdr}{\partial\Psi/\partial z}=\int\frac{dr}{[2R_0^2r^2-r^4-4\Psi/\nu^2]^{1/2}}=\int\frac{dr}{[(r^2-r_2^2)(r_1^2-r^2)]^{1/2}},$$

where $r_{1,2}^2/R_0^2=1\pm\sqrt{1-\Psi}$. We see that the specific volume and the "magnetic well" are expressed in terms of complete elliptic integrals:

$$V'(\Psi)=\frac{2K(k)}{r_2},\quad \frac{V''(\Psi)}{V'(\Psi)}=\frac{(2-k^2)^2}{8k^2}\left\{1+\frac{2-k^2}{k^2}\left[1-\frac{E(k)}{(1-k^2)K(k)}\right]\right\}, \tag{10}$$

where $k=\sqrt{1-r_2^2/r_1^2}$. At the magnetic axis, $r=R_0$, we have $\Psi=1$ and $k=0$. Correspondingly, we have $-V''/V'=3/16$. With $\gamma=5/3$, condition (9) yields $\beta>2.2$. At the boundary separatrix surface Σ we have $k\to 1$, $k'=\sqrt{1-k'^2}\to 0$, and $-V''/V'\to 1/8k'^2\ln(4/k')\to\infty$. A stability region thus arises near the magnetic axis

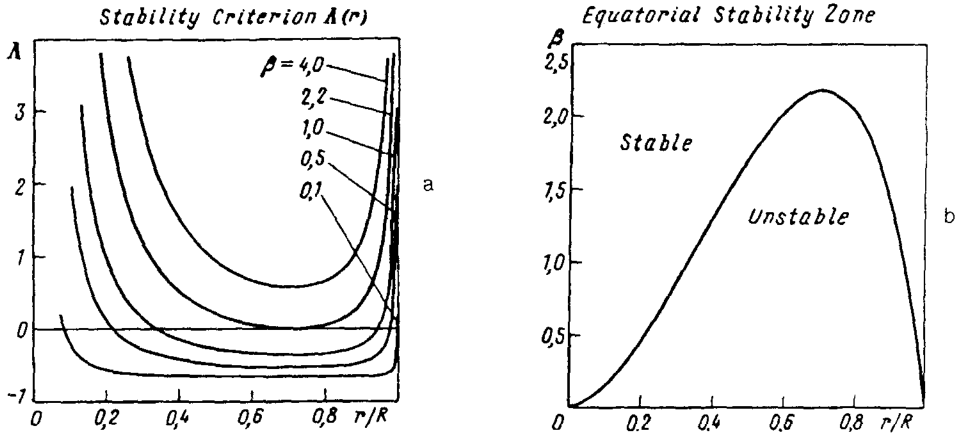

FIG. 2.

only at sufficiently large values, $\beta > 2.2$; at values $\beta < 2.2$, the only stable zone is near the boundary surface Σ. The width of the stable zone in this case tends toward zero as $\beta \to 0$ (Fig. 2).

3. Equilibrium and stability of plasmoids containing all three components of the current density

a. We restrict the discussion to the first meridional mode, $n=1$. According to (2), we write a solution which satisfies the boundary condition $\psi(R,\theta)=0$ at the surface of the sphere $r=R$ in the form

$$\psi=a_0\left[r^2-\frac{R^2 f_1(\alpha r)}{f_1(\alpha R)}\right]\sin^2\theta,\quad \psi_e=\frac{B_{0e}}{2}\left(r^2-\frac{R^3}{r}\right)\sin^2\theta, \tag{11}$$

where

$$a_0=-\frac{p'}{\alpha^2}=\frac{B_0/2}{1-\alpha^2R^2/3f_2\,(\alpha R)},\quad B_{0e}=\frac{2a_0\alpha R f_2(\alpha R)}{3f_1\,(\alpha R)}.$$

The pressure distribution inside the sphere $r=R$ is given by

$$p=p_0-\frac{p'^2}{\alpha^2}\Psi(r)\sin^2\theta,\qquad \Psi(r)=r^2-\frac{R^2 f_1(\alpha r)}{f_1(\alpha R)}. \tag{12}$$

In the case $f_2(\alpha R)=0$ the external magnetic field vanishes. The magnetic surfaces $\psi=\text{const}$ of the various radial modes constitute a system of nested tori separated by spherical separatrices. Figure 3 shows a configuration corresponding to the second radial mode, along with the pressure distribution $p(r)$ in the median plane $\theta=\pi/2$. In contrast with the preceding case, the pressure inside boundary sphere Σ can be either higher or lower than the external pressure. The pressure inside the plasmoid, averaged over the volume, is

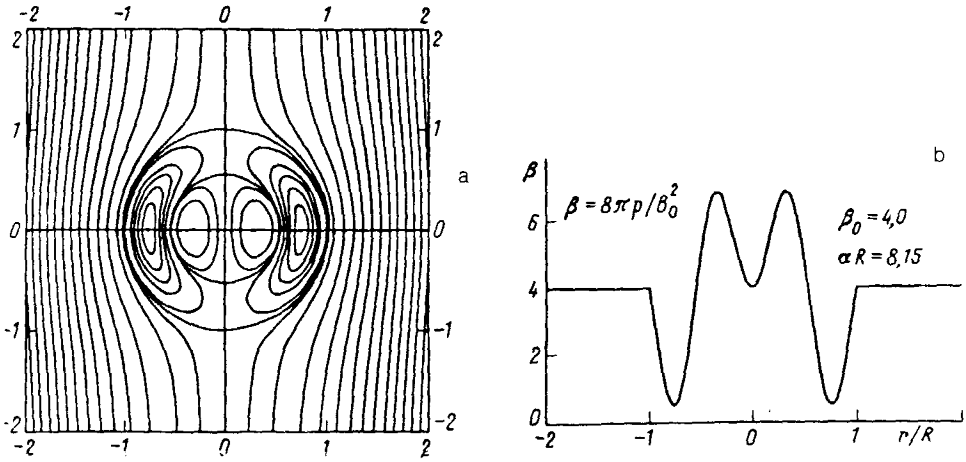

FIG. 3.

$$\langle p\rangle = p_0 - \frac{9f_1(\alpha R)B_{0e}^2}{4\pi f_2^2(\alpha R)}\left[1-\frac{5f_2(\alpha R)}{\alpha R f_1(\alpha R)}\right]. \tag{13}$$

This pressure, too, can be either higher or lower than the external pressure p_0. For the plasmoid in Fig. 3 we have $\langle p\rangle/p_0=0.56$. The existence of an equilibrium configuration with $p\geqslant 0$ requires $\beta=8\pi p_0/B_{0e}^2>3.7$.

A plasmoid which has a spheroidal boundary surface can be found by superimposing the first and third meridional modes, $n=1$ and $n=3$:

$$\psi=-\frac{p'}{\alpha^2}\left[r^2-a_1f_1(r)-a_3f_3(r)(1-5\cos^2\theta)\right]\sin^2\theta. \tag{14}$$

The coefficients a_1 and a_3 are found from the condition that the boundary surface $\psi=0$ pass through the circle $r=R$, $\theta=\pi/2$ and the points $r=z$, $\theta=0,\pi$:

$$a_1=[R^2f_3(z)+4z^2f_3(R)]/D, \quad a_3=[R^2f_1(z)-z^2f_1(R)]/D,$$

$$D=4f_1(R)f_3(z)+f_1(z)f_3(R).$$

For $z=R$ we thus have $a_3=0$, and we return to configuration (11) with a spherical boundary surface $r=R$.

b. Equilibrium configurations (11) and (14) refer to the general case, in which the Mercier necessary condition for stability is applicable.[6] If the equilibrium configuration is represented by the expansion

$$\psi-\psi_0=-\frac{R_0^2}{2}\frac{p'}{1+\nu^2}\left\{\left(1+c\frac{r^2-R_0^2}{R_0^2}\right)z^2+\frac{\nu^2}{4R_0^2}(r^2-R_0^2)^2-\frac{c-1}{12R_0^4}(r^2-R_0^2)^3\right\},$$

then Mercier stability of axisymmetric configurations near the magnetic axis, $r=R_0$, is determined by the condition[7]

$$p'(V)\left\{\frac{j_\varphi^2 R_0^2}{B_\varphi^2}-\frac{(\nu^2+1)^2}{\nu^4}\left[1+2\frac{1-\nu}{1+\nu}-\frac{1-\nu^2}{1+\nu^2}(1+2c)\right]\right\}>0. \qquad (15)$$

Here j_φ and B_φ are the azimuthal components of the current density and the magnetic field at the magnetic axis, $r=R_0$; $\theta=\pi/2$; and c is a characteristic of the triangularity of the cross sections of the magnetic surfaces near the magnetic axis. For a configuration with circular paraxial cross sections ($\nu=1$) and with a pressure which falls off with distance from the axis, $p'(V)<0$, we find the known limitation $j_\varphi R_0/B_\varphi<2$ from (15).

For equilibrium configuration (11), the position of the extrema, $r=R_0$, of the function $\psi(r,\theta)$ and the values of the parameters in (15) are expressed in terms of the functions $f_1(\alpha r)$ and $f_2(\alpha r)$ at $r=R_0$:

$$\alpha R^2 f_1'(\alpha r)/2rf_1(\alpha R)=1,$$

$$f_1'=\frac{2f_1}{\alpha r},\quad \nu^2=\frac{\alpha rf_1}{f_2}-1,\quad c=\frac{\alpha rf_1}{2f_2}-\frac{3}{2},\quad \frac{j_\varphi r}{B_\varphi}=\frac{2f_1}{f_2}.$$

The condition for the stability of configuration (11) can thus be expressed in terms of the ellipticity parameter $\nu=l_z/l_r$ of the cross sections of the magnetic surfaces near the magnetic axis:

$$p'(V)\left[\frac{4\nu^4}{\alpha^2 r^2}-\frac{3-\nu}{1+\nu}-\frac{(1-\nu^2)^2}{1+\nu^2}\right]>0. \qquad (16)$$

The derivative $p'(V)$ is positive at the minimum of the function $p(r,\pi/2)$ and negative at its maximum. For the plasmoid in Fig. 3, the configuration is unstable near the internal magnetic axis $r=R_{01}$, while it is stable near the second magnetic axis, $r=R_{02}>R_{01}$. Condition (15) shows that configurations with small values of ν are relatively stable in the case $p'(V)<0$. The corresponding small ratios l_z/l_r are evidently described by ψ function (14) in oblate ellipsoids. The conclusion that oblate "spheromaks" are relative stable agrees with a result derived in Ref. 8 for configurations which are approximately force-free, with $p'=0$.

Conclusion

The class of bounded axisymmetric equilibrium configurations analyzed here contains some configurations which have stable shells near the boundary surface. Exploiting the arbitrary nature of the density distribution, we might suggest that the maximum density is in the stable regions. It has thus been shown that bounded equilibrium configurations with an elevated density and, correspondingly, a depressed temperature in the shell adjacent to the separatrix boundary surface can exist. The helmet-shaped formations similar to rays in the solar corona fall in this category of configurations.[9] The spheroidal configurations in (6) and (11) show that an equilibrium Koutchmy plasmoid,[10] confined by the pressure of a uniform external magnetic field and by a plasma surrounding the spheroid, can exist in principle.

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