Fractional moments of distributions

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A new method is proposed for analyzing the multiplicity distributions of particles in inelastic processes. The method is based on the use of noninteger moments of distributions. In general, the method can be used to analyze any distributions which are encountered in physics or mathematics.

The purpose of this letter is to demonstrate the effectiveness of generalizing the concept of moments of distributions to moments of noninteger orders with the help of fractional-differentiation methods.[1,2] Although an analysis making use of fractional moments can be carried out for distributions which arise in any field of physics or mathematics, we will be stressing its application to multiple production of particles at high energies.

We assume a normalized probability distribution P_n ($\Sigma_{n=0}^{\infty} P_n=1$) In particular, this distribution could be the probability distribution for the occurrence of events with n particles in the course of multiple production. We define a generating function $G(z)$ for this distribution by

$$G(z)=\sum_{n=0}^{\infty}(1+z)^n P_n. \tag{1}$$

The factorial moments and cumulants of integer order q are given by

$$F_q=\frac{\Sigma_n P_n n(n-1)\ldots(n-q+1)}{(\Sigma_n P_n n)^q}=\frac{1}{\langle n\rangle^q}\frac{d^q G(z)}{dz^q}\bigg|_{z=0}, \tag{2}$$

$$K_q=\frac{1}{\langle n\rangle^q}\frac{d^q \ln G(z)}{dz^q}\bigg|_{z=0}, \tag{3}$$

where $\langle n\rangle=\Sigma_{n=0}^{\infty} P_n n$ is the average multiplicity. These moments and cumulants are related by the recurrence relations

$$F_q=\sum_{m=0}^{q-1}\frac{(q-l)!}{m!(q-m-1)!}K_{q-m}F_m. \tag{4}$$

Definitions (2) and (3) can be generalized to noninteger values of the order q by means of the formulas of (generalized) fractional differential calculus,[1] in which the role of the qth derivative is played by

$$D_z^q G(z)=\sum_{m=0}^{\infty}\frac{(1+z)^{m-q}\,G^{(m)}(-1)}{\Gamma(m-q+1)}. \tag{5}$$

Here $G^{(m)}$ are ordinary (integer) derivatives. We will make use of this generalized differentiation rule to calculate fractional moments of certain widely used distributions.

I. POISSON DISTRIBUTION

$$P_n=\frac{\langle n\rangle^n}{n!}e^{-\langle n\rangle} \quad \text{and} \quad G(z)=e^{z\langle n\rangle}. \tag{6}$$

Using (5), we find

$$F_q=\frac{e^{-\langle n\rangle}}{\langle n\rangle}\frac{\Phi(1,1-q;\langle n\rangle)}{\Gamma(1-q)}, \tag{7}$$

$$K_q=\frac{q}{\langle n\rangle^{q-1}\Gamma(2-q)}, \tag{8}$$

$$H_q\equiv\frac{K_q}{F_q}=\langle n\rangle e^{\langle n\rangle}\frac{q}{1-q}\frac{1}{\Phi(1,1-q;\langle n\rangle)}, \tag{9}$$

where Φ is the confluent hypergeometric function. At integer points we have $F_q=1$ and $K_q=H_q=\delta_{q1}$, as we should. On the intervals between integer points, both F_q and H_q oscillate. This oscillation can be seen in a particularly simple way in Eq. (8), on the basis of the behavior of the reciprocal of the gamma function. We also see that the amplitude of the oscillation falls off rapidly with increasing average multiplicity $\langle n\rangle$. At high multiplicities all the moments are nearly equal to their values at the integer points. Only the region $q>\langle n\rangle$, in which the oscillations are noticeable, could be of theoretical interest. However, essentially this entire region is difficult to reach experimentally because of the large measurement errors.

II. FIXED MULTIPLICITY

$$P_n=\delta_{nn_0}, \quad G(z)=(1+z)^{n_0}. \tag{10}$$

This distribution may be of interest if, for example, one singles out events which have a given multiplicity. Again using (5), we find

$$F_q=\frac{n_0^{1-q}}{\Gamma(1-q)}B(n_0,1-q), \tag{11}$$

$$K_q=\frac{n_0^{1-q}}{\Gamma(1-q)}[\psi(1)-\psi(1-q)], \tag{12}$$

$$H_q=\frac{\psi(1)-\psi(1-q)}{B(n_0,1-q)}, \tag{13}$$

where B is the Euler β function, and ψ is the logarithmic derivative of the gamma function. Here the transition to the values at integer points,[3] at which we have

$$F_q^{(i)}=n_0^{1-q}\frac{\Gamma(n_0)}{\Gamma(n_0-q+1)}=\frac{n_0(n_0-1)\dots\ (n_0-q+1)}{n_0^q}, \tag{14}$$

$$K_q^{(i)}=(-n_0)^{1-q}\,\Gamma(q)=(-n_0)^{1-q}(q-1)!\,, \tag{15}$$

$$H_q^{(i)}=(-1)^{1-q}n_0\,B(q,n_0-q+1)\,, \tag{16}$$

is particularly graphic. We see that the cumulants change sign at each successive integer q. The second part of (14) clearly reproduces (2). We also see that (14) is exactly the same as the analytic continuation of (11) to noninteger q values. On the other hand, expression (12) shows that a simple "analytic" continuation of cumulants (15) to noninteger q would be incorrect, although (12) does convert into (15) at integer values of q. To find the correct expression, we need to make use of Eq. (5). At high multiplicities n_0 we have $F_q \to 1$ and $K_q \to H_q \to 0$, as in the case of a Poisson distribution.

III. NEGATIVE BINOMIAL DISTRIBUTION

This distribution is one of primary interest, since it is often used to approximate measured distributions in the multiple production of particles:

$$P_n=\frac{\Gamma(n+k)}{\Gamma(n+1)\Gamma(k)}\left(\frac{\langle n\rangle}{k}\right)^n\left(1+\frac{\langle n\rangle}{k}\right)^{-n-k},\quad G(z)=\left(1-z\frac{\langle n\rangle}{k}\right)^{-k}. \tag{17}$$

From these expressions we can easily calculate the following:

$$F_q=\frac{(kx)^k}{\langle n\rangle^{q+k}}\frac{F(1,k;1-q;x)}{\Gamma(1-q)}, \tag{18}$$

$$K_q=\frac{k}{\langle n\rangle^q\Gamma(1-q)}\left[\frac{x}{1-q}F(1,1;2-q;x)+\ln\left(\frac{kx}{\langle n\rangle}\right)\right], \tag{19}$$

$$H_q=k\left(\frac{\langle n\rangle}{kx}\right)^k\frac{(x/1-q)\,F(1,1;2-q;x)+\ln\,(kx/\langle n\rangle)}{F(1,k;1-q;x)}. \tag{20}$$

Here F is the hypergeometric function, and $x=\langle n\rangle/(\langle n\rangle+k)$. For integer values of q we find the following[3] from (18)–(20):

$$F_q^{(i)}=\frac{\Gamma(k+q)}{k^{q-1}\Gamma(k+1)}, \tag{21}$$

$$K_q^{(i)}=\frac{\Gamma(q)}{k^{q-1}}, \tag{22}$$

$$H_q^{(i)}=\frac{\Gamma(q)\Gamma(k+1)}{\Gamma(k+q)}=kB(q,k). \tag{23}$$

These equations have been written in a form which allows the specification of a smooth interpolation between integer values of q and K. The actual expressions for the moments given by (18) and (20) also contains some oscillations, superimposed on this smooth interpolation. Unfortunately, expressions (18)–(20) are not very graphic and, in general, it is necessary to resort to numerical calculations in order to bring out the oscillations. However, in the limiting case of high multiplicities these oscillations can be demonstrated analytically. Specifically, in the limit $\langle n\rangle \gg 1$ it is a straightforward matter to show that the ratios of the actual moments to their interpolations in (21)–(23) have the behavior

$$F_q/F_q^{(i)} \to 1, \tag{24}$$

$$K_q/K_q^{(i)} \to 1 - \frac{k}{\langle n\rangle^q} \frac{\sin \pi q}{\pi} \ln \frac{\langle n\rangle}{k}, \tag{25}$$

$$H_q/H_q^{(i)} \to 1 - \frac{k}{\langle n\rangle^q} \frac{\sin \pi q}{\pi} \ln \frac{\langle n\rangle}{k}. \tag{26}$$

As in the case of the Poisson distribution, the oscillations of the noninteger moments die out with increasing multiplicity. This point can be seen from the behavior of the second term in the cumulants. It disappears at the integer points; between them it oscillates with an amplitude which decreases with increasing average multiplicity and with decreasing "number of sources" K. This behavior of factorial moments was first found in Ref. 2, through numerical calculations.

Consequently, these oscillations may find practical applications in research on multiple production only in studies of distributions with a large value of k and/or a small average multiplicity. Among such distributions are the distributions of particles in small volumes of phase space, in which interest has increased dramatically in connection with the concepts of intermittence and fractal systems (see the review[4]). A decrease in the average multiplicity is accompanied by an increase in the fluctuations in small intervals of the rapidity, and it is conducive for a study of the oscillations mentioned above. We thus acquire some new information about the phenomenon of intermittence. In general, an analysis of this sort may prove useful in those fields of physics which deal with large accumulations ($k \gg 1$) of weak ($\langle n\rangle \ll k$) independent sources. The results found in this case turn out to be extremely sensitive to the particular shape of the distributions and may serve as criteria for choosing the correct shape.

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