Asymptotic behavior of multicolor QCD at high energies in connection with exactly solvable spin models

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The contributions made by diagrams with an arbitrary number of reggeized gluons in the crossing channel to Regge processes of multicolor QCD are analyzed. The wave function describing bound states of gluons has the property of holomorphic factorization. The Hamiltonian for each of the two holomorphic subsystems is identical to the Hamiltonian for an exactly solvable lattice model which is a generalization of an isotropic Heisenberg magnet.

At high energies $\sqrt{s}$ in the leading-log approximation (LLA) $(g^2 \ln s \sim 1)$, we know[1] that a gluon is reggeized in quantum chromodynamics (QCD), while a pomeron, which is a bound state of two reggeized gluons, lies in the j plane of the t channel at the point $j=1+\omega$, where $\omega=(g^2/\pi^2)N \ln 2$ for the SU(N) gauge group. Although such a large intercept of a seed pomeron apparently agrees with recent experiments on deep inelastic ep scattering, the LLA result must be unitarized at sufficiently small values of the Bjorken variable x. One possible method for unitarization is based on an effective field theory for multi-Regge processes.[2] Another method, which is comparatively simple although not as versatile, corresponds to a generalization of the LLA in which diagrams with an arbitrary but conserved number of gluons in the crossing channel are taken into account.

It was shown in Ref. 4 that, within the framework of this method, the corresponding Bethe–Salpeter equation for n reggeized gluons simplifies considerably in the limit of a large number of colors, $N \to \infty$. In particular, the wave function $f_\omega(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n; \boldsymbol{\rho}_0)$, for the ground state $\psi_\omega(\rho_0)$ of gluons with coordinates $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n$ has the important property of holomorphic factorization in the 2D transverse $\boldsymbol{\rho}$ space:

$$f_w(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n; \boldsymbol{\rho}_0) = \sum_{\tau} f^{\tau}(\rho_{10}\, \rho_{20}, \ldots, \rho_{no}) \widetilde{f}^{\tau}(\rho_{10}^*\, \rho_{20}^*, \ldots, \rho_{n0}^*). \tag{1}$$

Here f^τ $(\widetilde{f}^\tau)$ are analytic (antianalytic) functions of the differences $\rho_{ij}=\rho_i-\rho_j$ $(\rho_{ij}^*=\rho_i^*-\rho_j^*)$ between the complex coordinates of the gluons. These functions satisfy the independent Schrödinger equations

$$\epsilon f^\tau = H f^\tau, \quad \widetilde{\epsilon}\widetilde{f}^\tau = H^* f^\tau, \tag{2}$$

where ϵ and $\widetilde{\epsilon}$ are corresponding "energies," in terms of which it is a simple matter to calculate the position of the singularities of the t-channel partial wave f_ω: $\omega=-(g^2/16\pi^2)\,N(\epsilon+\tilde{\epsilon})$. This position determines the asymptotic behavior $A(s,t)\sim s^{1+\omega}$ of the scattering amplitudes.

For a holomorphic subsystem, the Hamiltonian H is given by

$$H=\sum_{i=1}^{n} H_{i,i+1}, \tag{3}$$

where $i=n+1$ and $i=1$ are equivalent. The binary Hamiltonian $H_{i,i+1}$ can be written in two equivalent forms:[4]

$$\begin{aligned} H_{ik}&=P_i^{-1}\ln(\rho_{ik})P_i+P_k^{-1}\ln(\rho_{ik})+\ln(P_iP_k)-2\psi(1)\\ &=2\ln(\rho_{ik})+\rho_{ik}\ln(P_iP_k)\rho_{ik}^{-1}-2\psi(1), \end{aligned} \tag{4}$$

where $P_k=i(\partial/\partial\rho_k)$, $\psi(x)=\Gamma'(x)/\Gamma(x)$, $\psi(1)=-\gamma$, and γ is the Euler constant. It follows from (3) and (4) that the transposed operator H^T is found from Hamiltonian H [which corresponds to two different normalization conditions on the function f^T in Eq. (2)] through similarity transformations:

$$H^T=P_n^{-1}P_{n-1}^{-1}\dots P_1^{-1}HP_1P_2\dots P_n=\rho_{12}\,\rho_{23}\dots\rho_{n1}H\rho_{12}^{-1}\rho_{23}^{-1}\dots\rho_{n1}^{-1}. \tag{5}$$

It follows that there exists a differential operator A which commutes with H (Ref. 4):

$$A=\rho_{12}\,\rho_{23}\dots\rho_{n1}P_1P_2\dots P_n,\quad [A,H]=0. \tag{6}$$

It turns out[5] that there is a more general, mutually commuting system of operators $t(\theta)$:

$$t(\theta)=\mathrm{tr}\,T(\theta)=T(\theta)_{11}+T(\theta)_{22},\quad [t(v),t(u)]=0, \tag{7}$$

where $T(\theta)$ is the monodromy matrix in 2D spin space:

$$T(\theta)=L_{\rho 1}(\theta)L_{\rho 2}(\theta)\dots L_{\rho n}(\theta), \tag{8}$$

$$L_\rho(\theta)=\theta+\begin{pmatrix}\rho\partial & \partial\\ -\rho^2\partial & -\rho\partial\end{pmatrix}.$$

The coefficients of the expansion of $t(\theta)$ in (7) in a series in the spectral parameter θ include the operator A in (6) and the Casimir operator of the conformal group:

$$t_2=\sum_{i<j}\rho_{ik}^2P_iP_k. \tag{9}$$

Hamiltonian (3) and transfer matrix (7) are invariant under the conformal group of transformations,

$$\rho_i\to\frac{a\rho_i+b}{c\rho_i+d}, \tag{10}$$

with any complex parameters a, b, c, d.

Below we show that the operators in (7) commute with Hamiltonian (3), so the Schrödinger equations in (2), which describe the dynamics of the motion of n particles in a 2D plane, contain enough quantum conservation laws to be exactly solvable.

The property that causes the operators in (7) to commute with each other follows from the fact that monodromy matrix (8) satisfies the fundamental commutation relation[6]

$$T(v)_{i_1 i_1''} T(u)_{i_2 i_2''} (u-v+P_{12})_{i_1' i_2'}^{i_1'' i_2''} = (u-v+P_{12})_{i_1 i_2}^{i_1'' i_2''} T(u)_{i_2'' i_2'} T(v)_{i_1'' i_1'}, \tag{11}$$

where $(P_{12})_{i_1 i_2}^{i_1' i_2'} = \delta_{i_1 i_2}^{i_2' i_1'}$ is an index permutation operator. The matrix $L^{1/2}(\theta) = \theta + P_{12}$ is an L operator for an isotropic Heisenberg magnet, which is identical to the so-called XXX model.[6] The function $T(\theta)$ in (8) is interpreted physically as the monodromy matrix for spin systems with an interaction $\boldsymbol{\sigma} \cdot \mathbf{M}$, where the Pauli matrices $\boldsymbol{\sigma}$ couple neighboring indices on the edges of a 2D lattice in the horizontal direction, while the differential operator $\mathbf{M}$ ($M_3 = \rho\partial$, $M_- = \partial$, $M+ = -\rho^2\partial$) acts in the quantum subspace corresponding to vertical edges of the lattice. As the Hamiltonian H for Euclidian variables ρ_i in the quantum subspace we can choose any function of the commuting operators $t(\theta)$. A natural condition for this function is that the Hamiltonian be local, i.e., that there be an interaction of nearest neighbors, with coordinates ρ_i and ρ_{i+1}. A general method for constructing such a Hamiltonian, which commutes with $t(\theta)$ in (7), is known.[6] It is first necessary to construct a monodromy matrix in the form of a product of L operators which act in quantum and auxiliary spaces of the same dimensionality. In our case, the auxiliary subspace must be one-dimensional, so the L-operator is an integral operator which acts in the direct product of the quantum subspace (ρ_1) and the auxiliary subspace (ρ_2). This operator must have the following expansion in the spectral parameter θ:

$$L_{12}(\theta) = P_{12}\,[1 - \theta H_{12} + O(\theta^2)], \tag{12}$$

where the operator P_{12} permutes coordinates (ρ_1 and ρ_2), and H_{12} is the binary Hamiltonian which we are seeking. The total Hamiltonian H is expressed in terms of the sum of binary Hamiltonians $H_{i,i+1}$ in accordance with Eq. (3) (Ref. 6). To test whether H and $t(\theta)$ commute, it is thus sufficient to show that H_{12} in (4) is the same as that in (12). We know[6] that the operator L_{12} in (12) must satisfy a linear equation analogous to the fundamental commutation relation in (11):

$$L_{\rho_1}(v) L_{\rho_2}(u) L_{12}(u-v) = L_{12}(u-v) L_{\rho_2}(u) L_{\rho_1}(v), \tag{13}$$

where the operators $L_\rho(\theta)$ are written above [see Eq. (8)]. Equation (13) presupposes a multiplication of these operators as 2×2 matrices and their contraction with the unknown operators $L_{12}(\theta)$ along the coordinates ρ_1 and ρ_2. In particular, it follows from Eq. (13) that the generators of the conformal transformations in (10) commute with L_{12}, so H_{12} can depend on only the Casimir operator

$$\mathbf{M}_{12}^2 = -\rho_{12}^2 \partial_1 \partial_2 \tag{14}$$

of the conformal group. From matrix equation (13) we find the operator relation

$$\rho_{12}(\partial_1^{-1}-\partial_2^{-1})\rho_{12}^{-1}-[H_{12},\rho_{12}]=0. \tag{15}$$

Finally, it is a simple matter to verify that expression (4) for $H_{12}(\mathbf{N}_{12}^2)$ is, within an additive constant, a general solution of Eq. (16).

In summary, it has been demonstrated that the operators in (7) commute with each other and with Hamiltonian (3). It has been shown that the problem of solving Schrödinger equations (2) for a bound state of n reggeized gluons reduces to the problem of finding a representation of the Yang–Baxter algebra, (11). The standard method for constructing a representation of algebra (11) is based on the quantum version of the inverse scattering problem developed by Faddeev and his colleagues.[6] In this case, an additional difficulty in solving the problem stems from the noncompact nature of Möbius group (10).

We note in conclusion that the remarkable mathematical properties of Eqs. (2) can apparently be explained by the fact that the Yang–Mills theory is the low-energy limit of a structure theory which has a high symmetry group.

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