New type of solitary gravity wave at the surface of a deep liquid

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A new nonlinear integrodifferential equation describing gravity waves at the free surface of an ideal liquid of infinite depth is constructed in 2D hydrodynamics. Exact solitary-wave solutions of this equation are derived. The velocity field in these solutions is a strongly nonmonotonic function of the distance from the surface. The potential flow associated with such a wave contains moving local vortex singularities, both inside and outside the liquid. For irrotational flows, a generalization is made to the case of a liquid of finite depth.

1. Solitary gravity waves at the free surface of a liquid have been studied in detail only in shallow-water theory, in which the distribution of the velocity amplitude over thickness is approximately flat. A large number of nonlinear model equations have been proposed in that formulation of the problem. The most important are the Boussinesq and Korteweg–de Vries equations.[1] In the case of deep water, the research has focused on periodic waves of finite amplitude, their instability with respect to long-wave modulations, and the formation of envelope solitons as the end result of this instability.[2,3] These topics have been the focus of this research since the classic studies by Stokes. Again in the nonlinear theory, these waves are classical surface potential waves whose amplitude falls off monotonically (exponentially) with distance from the surface. No solitary waves in the form of single peaks or valleys in deep water have been derived theoretically so far, although motions of this sort at the surface of water, called layer bands, are well known in the open ocean.[4] In the present letter we show, in a rigorous, spatially 2D formulation of the problem, that steady-state solitary waves with a local vortex structure can exist near a free surface.

2. We consider the 2D problem of waves at the free surface of an infinitely deep, ideal liquid in a uniform gravitational field. We choose a Cartesian coordinate system whose Y axis is directed vertically upward and whose $y=0$ plane coincides with the unperturbed surface of the liquid. In the region of potential flow, the D^- potential $\phi(x,y,t)$ satisfies the Laplace equation with corresponding (generally nonlinear) boundary conditions $y=\eta(x,t)$ at the free surface.[1] We assume that the flow can contain isolated singular points of the pole type, so region D^- is, in general, multiply connected. We assume that the perturbations of interest are slightly nonlinear and have a length scale l:

$$|\phi_{x,y}| \ll 1, \quad |\eta| \ll 1. \tag{1}$$

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In the approximation quadratic in the amplitudes, we formulate a boundary-value problem for the half-plane $0<y<-\infty$ by moving the boundary condition away from the unknown surface $y=\eta(x,t)$ to the $y=0$ plane:

$$\phi_{xx}+\phi_{yy}=0, \quad x,y\in D^-, \tag{2}$$

$$\varphi_{tt}+\varphi_y+(\varphi_x^2+\varphi_y^2)_t-\varphi_t(\varphi_{tt}+\varphi_y)_y+...=0, \quad y=0, \tag{3}$$

$$\phi_{t|y=-\infty}=0. \tag{4}$$

An expression for the profile $\eta(x,t)$ is found from the Bernoulli equation:

$$\eta(x,t)=-\varphi_t+\varphi_t\varphi_{ty}-\frac{1}{2}(\varphi_x^2+\varphi_y^2)+\dots . \tag{5}$$

For the potential ϕ and its derivatives calculated at $y=0$, we change the notation to φ. In potential theory, the dimensionality of a boundary-value problem in terms of independent variables can be lowered by one by introducing a boundary value of the potential. An equation for this boundary value of the potential is a condition at the free surface.[5] For this purpose, we introduce the complex potential $W(z,t)=\phi(x,y,t)+i\psi(x,y,t)$ and the Keldysh function[6] $S(z,t)=iW_{tt}-W'$ in multiply connected region D^- of the plane $z=x+iy$. Here ψ is the stream function, t is a parameter, and the prime means the derivative with respect to z. If the limiting values of these quantities as $y\rightarrow -0$ are denoted by $W^-(x,t)$ and $S^-(x,t)$, then boundary condition (3) can be put in the form

$$\mathrm{Im}S^-(x,t)+(|W_x^-|^2)_t-\mathrm{Re}W_t^-ReS_x^-+...=0. \tag{6}$$

To find an equation for reconstructing the analytic function $W(z,t)$ in a multiply connected region $z\in D^-$, we take the approach of the linear conjugation boundary-value problem.[7] Using the Schwarz symmetry principle, we construct functions $W_*(z,t)$ and $S_*(z,t)$ which are analytic in region D^+ in the upper half-plane: $W_*(z,t)=\overline{W(\bar{z},t)}$ and $S_*(z,t)=\overline{S(\bar{z},t)}$. Their limiting values as $y\rightarrow +0$ are $\overline{W^-(x,t)}$ and $\overline{S^-(x,t)}$, respectively; the superior bar here means complex conjugation. Using the piecewise-analytic function $G(z,t)=S_*(z,t)$ for $z\in D^+$ and $G(z,t)=S(z,t)$ for $z\in D^-$, we then can write boundary condition (6) as a discontinuity in the function $G(z,t)$ at the crossing of the real axis:

$$G^+-G^-=2i[(|W_x^-|^2)_t-\mathrm{Re}W_t^-\mathrm{Re}S_x^-]. \tag{7}$$

In our case it is assumed that the function $W(z,t)$ and therefore $S(z,t)$ in the volume filled by the liquid have only isolated pole singularities, and that an infinitely remote point is an ordinary point for them. We write these functions as the sum of a regular part in the lower half-plane and a pole part: $W=W_R+W_P$ and $S=S_R+S_P$. The function $S_R(z,t)$, which is analytic in region D^-, is then reconstructed from the discontinuity in (7) and the poles with the help of Cauchy integral:

$$S_R(z,t)+\overline{S_p(\bar{z},t)}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(|W_\xi^-(\xi,t)|^2)_t-q(\xi,t)}{\xi-z}d\xi, \quad z\in D^-, \tag{8}$$

where $q(x,t) = \mathrm{Re} W_t^- \mathrm{Re} S_x^-$. Correspondingly, the limiting eigenfunctions $W^-(x,t)$ of homogeneous boundary-value problem (2)–(4) can be found from a solution of the nonlinear equation of dimensionality (1 + 1):

$$(W_R^- + \overline{W_P^-})_{tt} + i(W_R^- - \overline{W_P^-})_x + \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{(|W_\xi^-(\xi,t)|^2)_t - q(\xi,t)}{\xi - x + i0} d\xi = 0. \quad (9)$$

An initial-value problem for this equation can be formulated in exactly the same way as in a study of the motion of objects below the surface of a liquid, with the sources $W_P(z,t)$ assumed given.[8] For waves on deep water which are approximately linear ($W_{tt}^- \approx i W_x^-$), we can evidently use the approximation $q(x,t) = 0$. We call this model equation "truncated."

3. In the particular case in which the regular part of the complex potential W_R^- is a rational function with poles in the upper half-plane, we seek steady-state rational solutions of truncated equation (9) in the form of an expansion over the poles which is the same for the entire complex plane:

$$W_{(N_m)}^{(M)}(z,t) = \sum_{m=1}^{M} \sum_{n=1}^{N_m} \frac{C_{nm}}{(z - \lambda t - z_m)^n}, \quad z \in D^-. \quad (10)$$

Here $z_m = x_m + i y_m$ are the coordinates of the poles of order N_m in the complex plane, C_{nm} are their constant intensities, and λ is a spectral parameter of the problem. This parameter represents the velocity of the multiple-pole formation, which vanishes at infinity.

By virtue of the scale invariance of Eq. (9), we can choose the position of one of the poles in solution (10) arbitrarily. All the other unknown parameters can be found unambiguously—under the sole condition that there are no complex-conjugate pairs among the poles—from the system of $M + \Sigma_{m=1}^{M} N_m$ nonlinear algebraic equations

$$\lambda^2 (n-1) C_{n-1,m} - i C_{nm} + 2\lambda \sum_{m'=1}^{M} B_n^{mm'} = 0, \quad m = \overline{1,M}, \quad n = \overline{1, N_m + 1}, \quad (11)$$

$$B_n^{mm'} = \sum_{k=0}^{N_m + 1 - n} \sum_{k'=1}^{N_{m'}} \times \frac{(-1)^k \binom{k+k'}{k} k'(k+n-1)}{(z_m - \bar{z}_{m'})^{k+k'+1}} C_{k+n-1,m} \bar{C}_{k'm'}. \quad (12)$$

An exceptional property of nonlinear truncated equation (9) is that it has solutions of the form in (10) for *any natural values* of M and N_m. To find particular solutions of nonlinear algebraic system (11), we use a numerical method, specifically, Newton's method. Here we will discuss only one series of exact solutions corresponding to the complex potential:

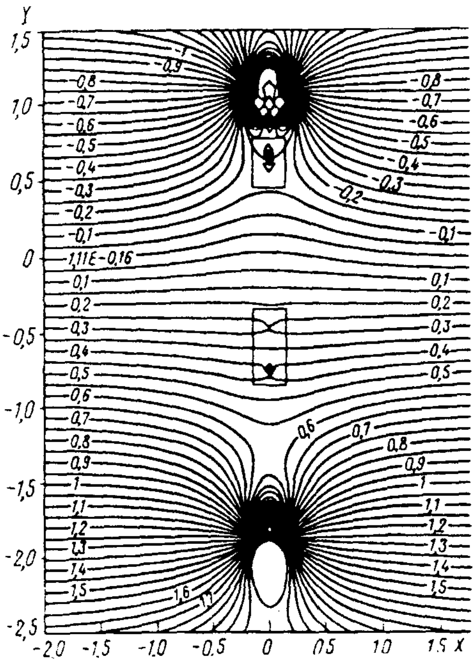

FIG. 1.

$$W^{(2M+1)}(z)=\frac{iC_{2,M}}{(z-\lambda t-iy_M)^2}+\sum_{m=-M}^{M}\frac{C_{1,m}}{z-\lambda t-iy_m},\quad z\in D^-. \tag{13}$$

All the poles lie on a common vertical line. They are numbered in such a way that the index increases with the height of the pole along the vertical. Poles with negative numbers and the zero pole are in the liquid ($y_{-m}=-y_m$), while those with positive numbers are outside it. The pole $m=M$ is a double pole, while the others are simple poles. For a given value of M, the corresponding system of algebraic equations in (11) is of $(4M+3)$th order. For values $M=1.17$, we studied a series of solutions corresponding to symmetric solitary waves with a single crest. It was found that the amplitude $\eta_{2M+1}(0)$ and the velocity λ_{2M+1} of the waves fall off monotonically as M increases.

Figure 1 shows a pattern of streamlines for $M=2$. This pattern was found as the imaginary part of the complex potential in the coordinate system of the wave: $\psi=-\lambda^{(5)}y+\mathrm{Im}\,W^{(5)}(x,y)$. The rectangles enclose simple, low-intensity poles with indices $m=-1$, 0, and 1. The null streamline corresponds, by definition, to the profile of the free surface. Figure 2 shows the same profiles, for the values $M=2$, 7, 12, and 17, as calculated from Eq. (5). In dimensional values, the parameters of the wave for the case $M=17$, $l=50$ m are a height ~ 0.8 m, a width ~ 500 m at the $y=0$ level, a velocity ~ 9 m/s, and a depth ~ 70 m for the lower pole. The condition that the nonlinearity be slight, (2), holds by a wide margin.

4. A large number of different multiple-pole solutions of nonlinear algebraic

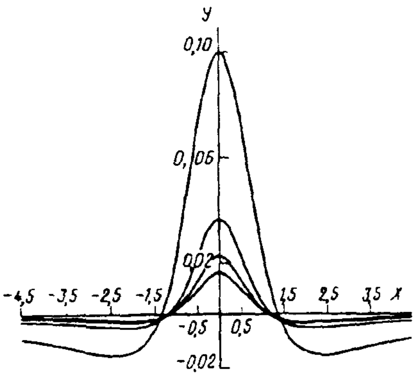

FIG. 2.

system (11) were calculated numerically, with one, two, and more maxima, for both the truncated equation and the complete model equation of the quadratic theory, (9). For an air–water two-layer medium, the ratio of the densities of the two layers is so small that the slower motions in the ocean are detected nearly instantaneously in the atmosphere. Accordingly, the existence of steady-state solitary waves in our model of an inertialess atmosphere can be thought of as the result of an interaction of potential flows in the atmosphere–ocean system. The exceptional diversity of these flows suggests that they may be primary energy sources and the factor responsible for the formation of storms (gales) and other anomalous states of the ocean surface.

5. In generalizing Eq. (9) to the case of a liquid layer of finite depth h, we restrict the discussion to purely potential flows.[9] The potential in a band is reconstructed unambiguously from the known function $\varphi(x,t)$ at $y=0$ and from the condition at the bottom, $\phi_{t|y=-h}$, by means of the integral representation

$$\phi(x,y,t)=\int_{-\infty}^{\infty}\varphi(\xi,t)g(\xi-x,y)d\xi, \tag{14}$$

$$g(x,y)=\frac{1}{h}\sin\frac{\pi y}{2h}\cosh\frac{\pi x}{2h}\left(\cos\frac{\pi y}{h}-\cosh\frac{\pi x}{h}\right)^{-1}. \tag{15}$$

Using (14) to calculate the necessary derivatives at $y=0$ in (3), we find, in the quadratic approximation for $\varphi(x,t)$, an explicit nonlinear integrodifferential equation:

$$\varphi_{tt}-L(\varphi_x)+\frac{\partial}{\partial t}[\varphi_x^2+(L(\varphi_x))^2]+\varphi_t[\varphi_{xx}-L(\varphi_{ttx})]=0, \tag{16}$$

where

$$L(\varphi(x,t))=\frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\xi,t)d\xi}{\sinh[\alpha(\xi-x)]},\quad \alpha=\frac{\pi}{2h}. \tag{17}$$

The integral in (17) is to be understood in the principal-value sense. Equation (16) is valid for arbitrary h; it takes dispersion effects into account exactly. In particular, when we use the relation $L(e^{ikx}) = i\tanh(kh)e^{ikx}$, we find from the linearized version of Eq. (16) the known dispersion relation for surface gravity waves: $\omega^2 = k\tanh(kh)$. Let us look at some limiting cases.

A. Shallow water ($h \to 0$)

The kernel of the L operator in (17) has a sharp peak at $\xi = x$. A power-series expansion of the function $\varphi(\xi,t)$ near this point yields, after an integration, $L(\varphi) \sim h\varphi_x + (h^3/3)\varphi_{xxx} + \ldots$. When we retain only the first of the nonlinear terms in (3), and replace it by the approximate value $(\varphi_x^2)_t \approx \mp(\varphi_x^2)_x$, where the $\mp$ correspond to the forward and backward waves, we find the Boussinesq equation,

$$\varphi_{tt} - h\varphi_{xx} - \frac{h^3}{3}\varphi_{xxxx} \mp 3(\varphi_x^2)_x = 0. \tag{18}$$

B. Deep water ($h \to \infty$)

According to (17), $L(\varphi)$ becomes a Hilbert transformation in this case:

$$\lim_{h \to \infty} L(\varphi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\varphi(\xi,t)d\xi}{\xi - x} \equiv H(\varphi). \tag{19}$$

Introducing the limiting functions $W^{\pm}(x,t) = \varphi \pm iH(\varphi)$, we can put Eq. (16) in the form in (9) if there are no pole singularities ($W_P = 0$).

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