Possibility of observing an anomalous chaos-to-regularity transition

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An anomalous chaos-to-regularity transition may occur in simple Hamiltonain systems. The existence of such a transition is demonstrated in the example of an anharmonic oscillator subjected to a monochromatic external field.

Motivation for studying the behavior of very simple nonlinear systems in a monochromatic external field is that experimental results[1] can be prepared with theoretical predictions.[2]

An external field which is periodic in time induces a dense set of resonances in the phase space of a nonlinear conservative Hamiltonian system. The positions of these resonances, I_k, are determined by the condition for a resonance between the normal frequencies $\omega(I)$ (I is the action variable of the unperturbed Hamiltonian) and the frequency of the external perturbation, Ω. If the external field is sufficiently weak, the initial resonances remain isolated. As the amplitude (F) of the external field is raised, the widths (W_k) of the resonance zones increase, and at $K > F_{\text{cr}}$ they overlap. When this overlap occurs, i.e., under the condition

$$\bar{W}_k = \Delta I_k, \tag{1}$$

where $\bar{W}_k = \frac{1}{2}(W_k + W_{k+1})$, and $\Delta I_k = |I_k - I_{k+1}|$, it is said that there is a transition to a global stochastic behavior in the corresponding region of the phase space. Condition (1) for the overlap of Chirikov resonances[3] provides a reliable estimate of the critical level of the perturbation, F_{cr}, i.e., the level required for a transition to a global stochastic behavior.

The behavior of the widths of the resonances, $\bar{W}_k$, and the distances between them, ΔI_k, as a function of the resonance index is simplest when the satisfaction of resonance overlap condition (1) for index k_1 (at a fixed level of the external perturbation) guarantees that this condition holds for arbitrary $k > k_1$. This is precisely the situation which prevails in the extensively studied systems of a 1D Coulomb potential[4] and a square well,[5] subjected in each case to a monochromatic perturbation. In the former case we have $\bar{W}_k \approx k^{1/6}$ and $\Delta I_k \approx k^{-2/3}$, while in the latter we have

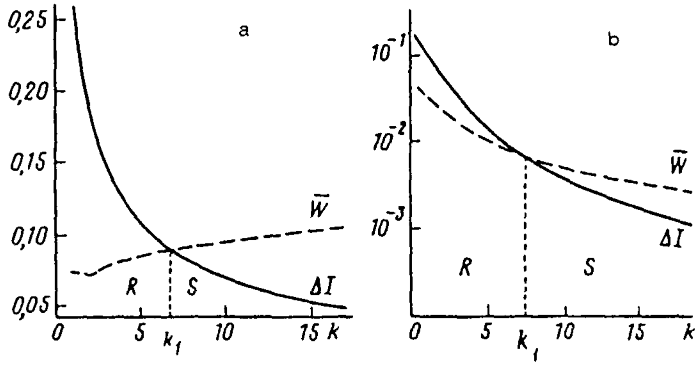

FIG. 1. The widths of the resonances, $\bar{W}_k$, and the distances between them, ΔI_k, versus their index. a—For a 1D Coulomb potential; b—for a square well. The region of k values to the left of k_1 is regular (R), while that to the right is stochastic (S), since the width of the resonances is greater than the distance between them.

$\bar{W}_k \approx k^{-1}$ and $\Delta I_k \approx [k(k+1)]^{-1}$. As can be seen from Fig. 1, for both the 1D Coulomb problem and the square well there is a regularity–chaos transition (we will call this a "normal" transition), since there exists a unique point k_1 such that at $k > k_1$ the condition $\bar{W}_k > \Delta I_k$ always holds. The motion is therefore chaotic. However, as the behavior of the widths of the resonances and of the distances between them as a function of the index of the resonance becomes more complex, we can allow the appearance of an additional intersection point and thus a new transition: a chaos–regularity transition, which we will call "anomalous." There is also the exotic possibility of the intermittent occurrence of regular and chaotic regions in phase space.

In this letter we demonstrate that an anomalous chaos–regularity transition occurs in a simple Hamiltonian system: an anharmonic oscillator subjected to a monochromatic perturbation. The dynamics of the system is generated by the Hamiltonian

$$H(p, x, t) = H_0(p, x) + Fx\cos\Omega t, \qquad (2)$$

where the unperturbed Hamiltonian is

$$H_0(p, x) = \frac{p^2}{2m} + Ax^n = E(n = 2l,\ l > 1). \qquad (3)$$

System (3) fills a gap between two extremely important physical models: the harmonic oscillator ($n=2$) and the square well ($n=\infty$).

System (2) has recently been studied,[6,7] but only for amplitudes F well above the values required for an overlap of neighboring resonances. A fairly detailed bibliography on this subject is given in Ref. 7.

In terms of action–angle variables (I, θ), the Hamiltonian $H_0(p, x)$ becomes

$$H_0(I) = \left[\frac{2\pi}{\alpha G(n)} I\right]^{\alpha}, \qquad (4)$$

where

$$G(n)=\frac{2\sqrt{2\pi m}\Gamma(1+\frac{1}{n})}{A^{1/n}\Gamma(\frac{1}{2}+\frac{1}{n})}, \quad \alpha=\frac{2n}{n+2}.$$

The resonant values of the action, I_k, found from the condition $k\omega(I_k)=\Omega$, are

$$I_k=\frac{2n}{n+2}\left(\frac{G(n)}{2\pi}\right)^{2n\beta}\left(\frac{\Omega}{k}\right)^{(n+2)\beta}, \quad \beta=\frac{1}{n-2}. \tag{5}$$

A classical analysis based on the resonance-overlap criterion leads to the following expression for the critical amplitude of the external perturbation:

$$F_k^{cr}=2^{-(3n-4)\beta}\frac{n(n-2)}{(n+2)^2}\frac{1}{x_k}\left[\frac{G(n)}{\pi}\right]^{2(n-1)\beta}\Omega^{2(n-1)\beta}k^{6\beta}$$
$$\times[k^{(2+n)\beta}-(k+1)^{(2+n)\beta}]^2, \tag{6}$$

where x_k is a Fourier component of the coordinate $x(I,\theta)$. Expression (6) solves the problem of reconstructing the structure of the phase space of this Hamiltonian for arbitrary values of the parameters. Here is brief list of the basic features of this structure:

1. For a square well ($n=\infty$) and for arbitrary F, one can specify an energy (or the index of a resonance) above and below which the motion is regular and stochastic, respectively.
2. For any $n<\infty$ one can always specify a value of the external perturbation, $F_0(n)$, such that at $F<F_0(n)$ the motion remains regular at all energies.
3. For any $F>F_0(n)$ one can specify an energy interval within which the motion is chaotic, while outside the interval the motion is regular.
4. For a given value of n, the size of the chaotic interval depends on only the amplitude of the external field, not on the frequency.

The effect found here can be observed in the case $n=4$, which has often been discussed in the literature. If the parameters of the Hamiltonian have the values $A=m=\Omega=1$, the anomalous chaos–regularity transition for the resonance $k=5$ (for example) occurs when the external field has an amplitude $F_{cr}\sim 0.029$. In this case we have $F_0(4)=0.011$. The critical amplitudes for real systems, i.e., for arbitrary values of the parameters of the Hamiltonian, can be found through a simple scaling. At $n\geqslant 8$ the structure of the phase space depends smoothly on the degree of anharmonicity. We will thus use the results of an analysis of the motion for $n=8$ as an illustration.

The "phase diagram" in Fig. 2 can be used to determine, at a fixed level of the external perturbation, the energy intervals of regular and chaotic motion. The snapshot of $E(x)$ at the right in Fig. 2 confirms that an anomalous chaos–regularity transition occurs. We can clearly see isolated nonlinear resonances which persist at large values of k, and near which the motion remains regular. The reason for this

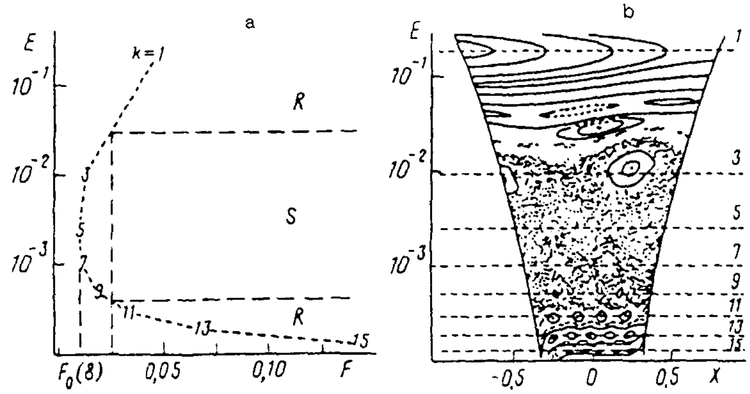

FIG. 2. a: "Phase diagram," i.e., critical values of the amplitude of the external field for various resonance indices (specified by the numbers 1–15 on the curve). The upper part of this curve ($k=1$–5) corresponds to the normal transition, and the lower part ($k=5$–15) to the anomalous transition. b: Snapshot of the trajectories of Hamiltonian (2) against the background of an x^8 potential well for $F=0.022$ ($A=1$, $m=1$).

anomaly is explained by Fig. 3. The plots of the resonance widths and of the distances between resonances in this figure demonstrate that there are two intersection points, rather than one; these points are $k=k_1$ and k_2. Consequently, there is an anomalous chaos–regularity transition.

One of the most important consequences of a transition to a global stochastic behavior is a diffusive energy transport. For systems in which an anomalous chaos–regularity transition is possible, any chaotic interval is bounded both above and below by regular intervals. As a result, there is a limitation on the diffusive increase in (or diffusive loss of) the energy of a particle. The anomalous transition thus results in a stabilization of the particle in a certain energy interval. In particular, stochastic diffusion occurs for Rydberg states of an electron in a hydrogen-like atom.[1,2] For semiclassical states which are described well by a 1D Coulomb potential, an electron has a tendency to diffuse toward higher energies, toward the point of ionization. However, a deviation of the potential from a Coulomb potential (due to screening, for example) may lead to the onset, at high energies, of an anomalous chaos–regularity transition and to the localization of the electron in a finite energy interval.

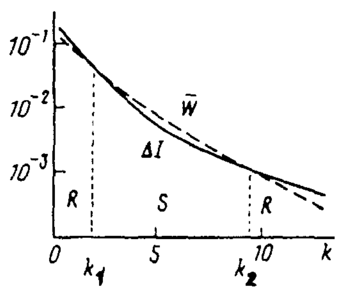

FIG. 3. The same as in Fig. 1, but for an x^8 potential. The anomalous transition occurs near the 10th resonance.

Numerical simulations have shown[8] that system (2) can be used to describe rotational excitations of diatomic molecules, e.g., CsI, subjected to an intense electromagnetic field. In such experiments an anomalous regularity–chaos–regularity transition can be seen in changes in the statistical properties of the quasienergy spectra and also in the structure of the quasienergy wave functions.

The distinctive features of the phase space which we have found here might be manifested in experiments on IR absorption in quantum dots[9] consisting of a 2D electron gas confined in the plane of a heterojunction of a semiconductor by an anharmonic potential.[10]

Finally, as a suitable entity for observing this effect we might suggest Paul atomic traps,[11] which are described well by a 1D model of a charged particle in an anharmonic static potential subjected to a monochromatic dipole perturbation.

In conclusion we wish to point out that the structure of the phase space of this 1D system with a periodic perturbation closely resembles that of a 2D conservative Hamiltonian system with a compact region of a local instability. In each case there is a anomalous regularity–chaos–regularity transition. In the former case, as we have shown in this letter, the transition occurs as the amplitude of the external field is varied; in the latter case,[12] it occurs as a parameter of the nonlinearity of the corresponding 2D potential is varied.

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