Stochastic resonance and self-organization in a parametric interaction

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A parametric acoustic–electromagnetic interaction in the presence of a small noise signal is analyzed. Under supercritical conditions, a self-organization occurs in the system. This self-organization consists of the formation of a spatial–temporal structure: a bound pair of large-amplitude acoustic solitons which move through the crystal periodically.

Effects associated with a stochastic resonance and self-organization have recently been drawing considerable attention in research on noisy systems (see Ref. 1 and the literature cited there; see also the recent review[2]). The topic of primary interest here is the active role played by the noise in the excitation of coherent motion. In a typical case, the response of a nonlinear multistable system to a periodic external force in the presence of a noise is studied. However, there is much interest in nonlinear systems in which, in the absence of an external force, a periodic motion is excited as a result of the onset of an instability which leads to a transition to a new, nonuniform state.[3]

We consider an anisotropic crystal which has a nonlinear (quadratic) electrostriction and in which two parallel electromagnetic waves and one acoustic wave are propagating and interacting. It is assumed that the frequencies and wave vectors satisfy the matching conditions $\omega_1-\omega_2=2\Omega$ and $k_1-k_2=2K$. For supercritical perturbations in an infinite medium, an interaction of this sort gives rise to a soliton acoustic-wave pulse. The velocity of this pulse is on the order of the sound velocity and is set by a balance struck between parametric amplification and depletion of the source, which tends to reduce this amplification.[4]

For a bounded crystal of length l, we need to consider the reflection of sound from the boundaries. This reflection leads to a repeated passage of the pulse through the interaction region and to a buildup of nonlinear effects. In the simplest case, discussed below, only the forward acoustic wave interacts actively with the electromagnetic pump wave, because of the matching conditions.

The dimensionless equations for the wave amplitudes, $\partial A_{1,2}/\partial z=\mp A_{2,1}A_3^2$, $(\partial/\partial\tau+\partial/\partial z+\Gamma)A_3=A_1A_2A_3$, can be simplified by the substitutions $A_1=\cos\phi$, $A_2=\sin\phi$, $A_3^2=\partial\phi/\partial z$. After the reflected wave is eliminated, these equations lead to the following closed equation for ϕ:

$$(\partial/\partial\tau+\partial/\partial z+2\Gamma)\phi=\sin^2\phi+[f_3(\tau)+R^2(\partial\phi(L,\tau-L)/\partial z)^{1/2}]^2, \qquad (1)$$

where $R^2=R_0R_L\exp(-\Gamma L)$, and R_0 and R_L are the reflection coefficients at $z=0$ and L. The initial and boundary conditions are $A_1(0,\tau)=1$, $A_2(0,\tau)=0$, $A_3(0,\tau)=f_3(\tau)+R^2A_3(L,\tau-L)$, $A_3(z,0)=0$, and $A_3(L,\tau)=0$ for $\tau<0$. Here

$L = l/l_{\text{nl}}$, l_{nl} is a length scale of the nonlinear interaction, and Γ is an attenuation parameter. Working from (1), we can write the equation for the total energy of the forward acoustic wave,

$$W(\tau) = \int_0^L A_3^2(z,\tau)\,dz,$$

in the form

$$dW/d\tau = -2\Gamma W + \sin^2 W + [f_3(\tau) + R^2(\partial\phi\,(L,\tau - L)/\partial z)^{1/2}]^2 - \partial\phi\,(L,\tau - L)/\partial z. \qquad (2)$$

For an infinite medium ($L \to \infty$ and $|f_3| \ll 1$) the dynamics of the soliton which forms is determined by the spectrum of steady-state values of the energy, $W = W(\Gamma)_i$, where W_i satisfy the equation $2\Gamma W = \sin^2 W$. This equation has either one root or several (an odd number), depending on the parameter Γ (which is a control parameter). The first nonvanishing solution arises at the value $\Gamma = \Gamma_c \approx 0.36$, which is the point of the first bifurcation of (1) (Ref. 4). Denoting the state with $W=0$ as W_0, we see that the even and odd roots correspond to respectively stable and unstable steady-state values of the energy (W_1 is the first unstable value). An instability occurs under the two conditions $\Gamma < \Gamma_c$ and $W_1 < W$.

When the boundedness of the medium (L) and the reflection are taken into account, the picture becomes much more complicated. A soliton which forms does not leave the interaction region. Being reflected into a backward wave, it returns to the entrance after a delay. Because of absorption ($\Gamma \neq 0$) and the nonideal nature of the

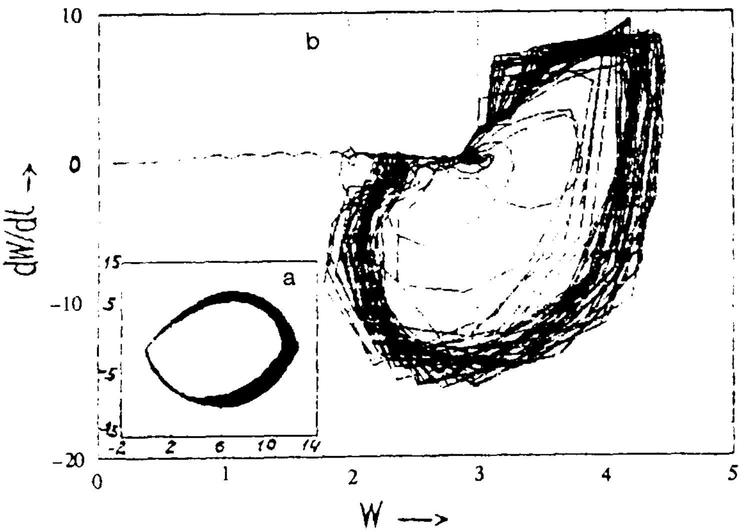

FIG. 1. The $(dW/dt, W)$ phase plane. a—Linear regime; b—nonlinear regime.

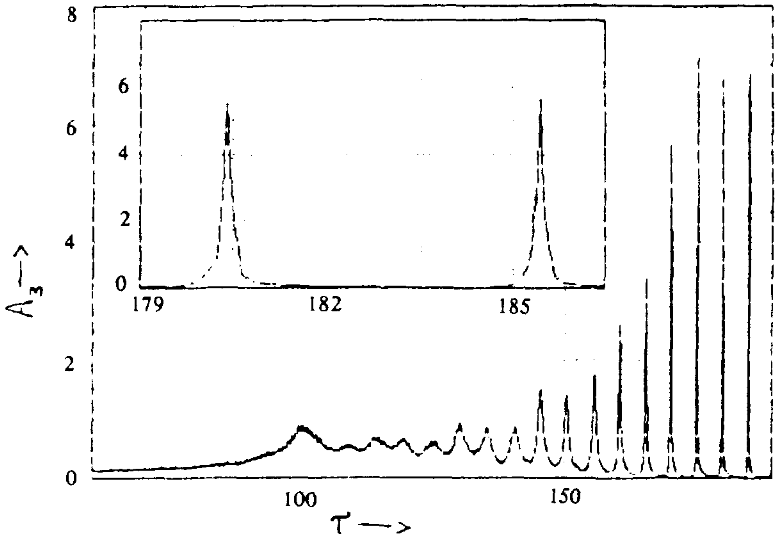

FIG. 2. Time evolution of the amplitude of the sound at the exit, $A_3(L,\tau)$.

reflection ($R_{0,L}<1$), there is a dissipation of energy. As a result, the dynamics becomes complex: The rereflected pulse reproduces neither a steady-state shape nor a steady-state energy. Because of the complexity of the problem, we carried out a numerical simulation, specifically, a numerical solution of Eq. (1) with various parameter values. The calculations were carried out in a second-order scheme, whose accuracy was monitored by reducing the step of the mesh. The results shown in the figures correspond to the parameter values $L=5$, $\Gamma=0.015$, $R_0R_L=0.9$, a time-mesh step $\Delta\tau=0.023$, and an input-signal amplitude $f_{30}=0.05$.

The integral time evolution of the process is useful to analyze in the $(dW/d\tau, W)$ phase plane with the help of Eqs. (2) and (1). Figure 1a shows the phase plane in the absence of pumping, with $R=1$, $\Gamma=0$, and a pulsed input signal $f_3(\tau)=\exp[-(\tau-2)^2]$. We see that the trajectory has the standard closed-curve form, corresponding to a linear sound pulse which is rereflected with a period $T=2L$. When there is damping, the trajectory contracts toward the origin of coordinates. This contraction corresponds to a relaxation of excitations in the crystal.

When a noise signal is present, the dynamics becomes quite different. {In the calculations, a noise $f_3(\tau)$ was simulated by generating a random quantity distributed uniformly along the interval $[0, f_{30}]$.} The calculations show that under subcritical conditions only the noise is observed at the exit from the crystal. The Fourier spectra of the input and output signals are essentially identical in this case. When the conditions become even slightly supercritical, in contrast, the output signal begins to grow rapidly, and a high-power soliton-like acoustic pulse forms in the crystal (Fig. 2). Over a time on the order of several passes, the pulse is amplified sharply, and it contracts. However, these processes occur in such a way that the energy of the pulse,

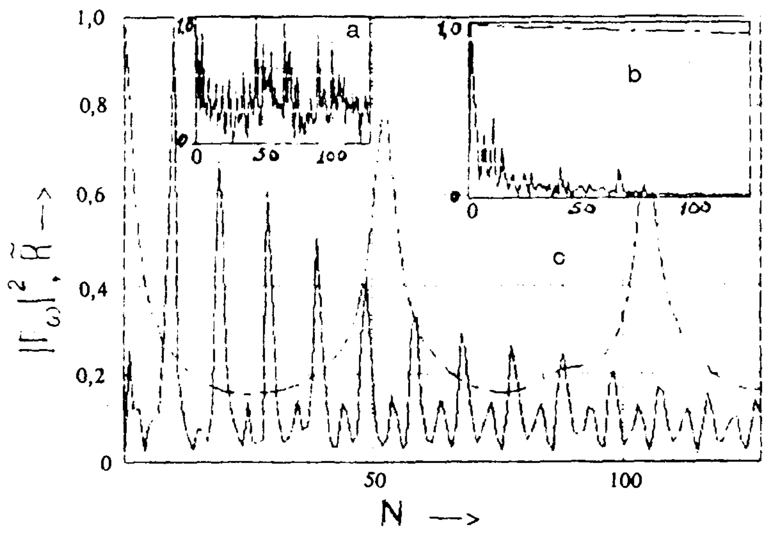

FIG. 3. Fourier spectrum $|F_\omega|^2$ (solid curve) and autocorrelation function $\tilde{R}$ (the dot-dashed curve) of the sound amplitude at the output, $A_3(L,\tau)$, as a function of the index (N) of the temporal-sampling point. a—Linear regime; b—slightly nonlinear regime (initial sampling); c—very nonlinear regime.

W, approaches W_2. The Fourier spectrum of the output signal consists of many sharp peaks. The autocorrelation function clearly indicates the existence of a coherent motion in the system (Fig. 3). Significantly, the period of the motion is equal to half the time required to traverse the entire crystal: $T=L$. Over the observation time, the output signal evolves from small-amplitude noise. At the time of generation it acquires the typical form, a periodic train of giant soliton-like pulses. The width of the pulses quickly becomes much shorter than the length of the crystal (Fig. 2).

A state of this sort probably appears on the basis of the following scenario. Because of the undamped source of noise at the input, and because of rereflection from the boundaries, there is a progressive buildup of energy in the system. If $W > W_1$, an instability occurs in the system; as a result of the development of this instability, a soliton forms. However, after the soliton is reflected into a backward wave, the threshold conditions which lead to soliton creation are satisfied again. As a result, a second soliton is nucleated in the crystal. Since the time at which the generation of the pulse which is formed is not determined, and since the rate at which the crest of this pulse moves is below the sound velocity, the two solitons begin to interact strongly after the arrival of the first pulse. The nature of this interaction depends on the position of the total energy W in the steady-state spectrum. In the first stage, the result is an extremely complex dynamics, with the phase trajectories becoming entangled near the points $W_{2,3}$. However, because of energy dissipation, a new limiting cycle is drawn in the system, after a fairly long transient process (Fig. 1b). This circumstance leads to the onset of periodicity in the motion of a bound soliton pair, which "runs through" the crystal. Since each pulse is unsteady, this reproducibility becomes possible if the pulses do not overlap and if the on-center distance between them is equal to the length of the crystal. The average energy of the pair, W (which can be thought of as the energy of the coherent motion), varies slightly as the noise amplitude f_{30} is increased over the interval 0.05–0.5. If we take the ratio W/f_{30}^2 to be the signal-to-noise ratio, then we conclude that a stochastic resonance can occur in this medium without a periodic external force. It thus becomes possible to extend the concept of a stochastic resonance not only to autonomous systems with a limiting cycle[1] but also to unstable media. In such media, a coherent state can develop as a response to a noise perturbation. It corresponds to a self-organization, which is manifested in the formation of a coherent, nonuniform, time-varying structure.

As the conditions become more supercritical (as Γ is reduced further), higher steady-state points W_i come into play, with the result that the spatial and temporal scales of the structure decrease. The motion becomes more complex, according to the calculations. Under very supercritical conditions, the pulses begin to break up. Because of the large number of steady-state points, the phase curve becomes greatly entangled, and the process itself acquires chaotic features.

These results reveal a self-organization in a comparatively simple, nonlinear model. In addition, from the experimental point of view, they demonstrate that a train of ultrashort sound pulses can be generated in an anisotropic crystal.

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