Radiative corrections to the pion beta decay

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The QED radiative corrections to the width of the pion β decay are calculated by choosing muon decay as the normalization process. The hardonic effects in pion–virtual photon interactions are taken into account using the vector meson dominance model. The uncertainties due to the strong interactions and the contribution from the background process were found to be small. The pure pionic radiative correction $\delta_\pi = -1.5\%$ was obtained.

The pion β decay was initially discussed in 1954 in a paper by Ya.B. Zeldovich and observed for the first time 30 years ago.[1] Recently, an interest in this rare decay (its branching ratio is $Br \sim 10^{-8}$) was rekindled. The aim of the experiments of Ref. 2 is the precise measurement of the V_{ud} matrix element of the Cabbibo–Cobayashi–Maskawa (CKM) matrix. In terms of the sum of squares of the first row in the CKM, the present experimental results for the neutron β decay are $2.2\times\sigma$ higher and those from the superallowed nuclear β decay are $2.5\times\sigma$ lower[3] than the standard model (SM) prediction (for three generations of quarks)—unity. The discrepancy of the average values of V_{ud} extracted from those experiments exceed 1%. Although the accuracy is better than 0.5%, it is necessary to solve, for example, the problem of the possible existence of an additional b' quark (if $V_{ub'} \neq 0$). The pion beta decay is theoretically more clean than the nuclear and nucleon beta decays, where a series of additional factors must be taken into account. To absorb the large radiative corrections connected with the Fermi coupling constant, we use the muon decay as a "normalization" process.

The radiative corrections (RC) to the pion beta decay width in the SM were calculated by Sirlin.[4] The superallowed nuclear $0^+ \to 0^+$ transitions were considered as the normalization process. The pure pionic correction $\delta_\pi = +3.1\%$ was obtained. In addition to the other choice of the normalization process, the difference between this result and our result, $\delta_\pi = -1.5\%$, could follow from the different approaches used to take the strong interaction effects into account. The scale of distances where the strong interaction is essential is much larger than that of the weak interactions. The loop integrals containing the photon-pion vertex converge at the loop momenta on the order of the ρ-meson mass—the characteristic scale of strong interactions. To describe the strong interaction effects due to a coupling of a virtual photon with a momentum k with a pion, we use the vector meson dominance model (VDM), which leads to the following substitution for a photon propagator:

$$\frac{1}{k^2} \to \frac{m_\rho^2}{k^2(m_\rho^2-k^2)} = \frac{1}{k^2} + \frac{1}{m_\rho^2-k^2}\,. \tag{1}$$

This substitution is equivalent to the following choice of the cutoff parameter in the old-fashioned weak interaction theory: $\Lambda = m_h = m_\rho$. Uncertainty due to the choice of a characteristic hardronic mass m_h in the value of the radiative corrections is on the order of $\alpha/\pi \ln m_\rho/m_\phi \leq 1\%$.

In Sec. 1 of this paper, we calculate the contribution of a radiative pion beta decay. In Sec. 2, some other one-loop radiative corrections are considered. In this section, we discuss some sources of uncertainties due to strong interactions and estimate the contribution of the background process, which was found to be small.

The width of the pion β decay in the Born approximation has the form[5]

$$\Gamma_0 = \frac{G^2\Delta^5|V_{ud}|^2}{\pi^3}\left(1-\frac{\Delta}{2m_\pi}\right)^3 I(\mu), \tag{2}$$

$$I(\mu) = \int_\mu^1 dx x^2(1-x)^2\beta$$

$$= \frac{1}{30}\left[\left(1-\frac{9}{2}\mu^2-4\mu^4\right)\sqrt{1-\mu^2} + \frac{15}{2}\mu^4 \ln\frac{1+\sqrt{1-\mu^2}}{2}\right],$$

$$\beta = \sqrt{1-\frac{\mu^2}{x^2}}, \tag{3}$$

$$\mu = \frac{m_e}{\Delta}, \quad \Delta = m_{\pi+} - m_{\pi 0} \approx 4.59 \ \text{MeV}.$$

The large electroweak corrections arising from the W boson self-energy are the same as those for the muon decay width. The ratio of the π decay width and the muon decay width is free from the electroweak corrections:

$$\frac{\Gamma(\pi^+ \to \pi^0 e^+ \nu_e)}{\Gamma(\mu^+ \to \bar{\nu}_\mu + \nu_e)} = \frac{|V_{ud}|^2 192}{30}\left(\frac{\Delta}{m_\mu}\right)^5\left[1+\frac{\alpha}{2\pi}\left(\pi^2-\frac{25}{4}\right)\right]$$
$$\times\left(1+\delta_\pi\left(1-\frac{3\Delta}{2m_\pi}-\frac{5m_e^2}{\Delta^2}\right)\right). \tag{5}$$

In the expression for the muon width we omitted the contributions of the box-type diagrams with the W and the Z bosons. These contributions do not exceed 0.01% (Ref. 6).

1. Radiative $\pi\beta$ decay

Let us consider the QED radiative corrections to the width of the $\pi+\beta$ decay

$$\pi^+(p_1) \to \pi_0) + e^+(p_e) + \nu(p_\nu) + \gamma(k). \tag{5}$$

The differential width of the radiative decay (Fig. 1) has the form

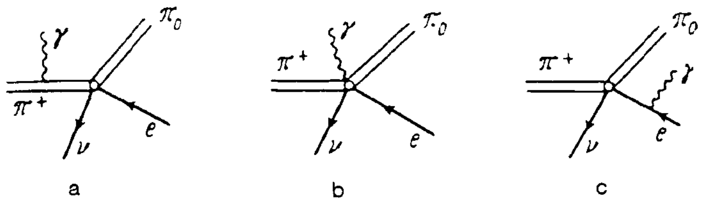

FIG. 1. The Feynman diagrams for the radiative $\pi\beta$ decay.

$$d\Gamma = -\frac{\alpha}{4\pi^2}\frac{d^3k}{\omega}d\Gamma_0(2\varepsilon_\nu\varepsilon_e - p_\nu p_e)^{-1}\left\{(2\varepsilon_\nu\varepsilon_e p_\nu p_e)\left[\frac{1}{\omega^2}+\frac{m_e^2}{(p_e k)^2}-\frac{2\varepsilon_e}{\omega(p_e k)}\right]\right.$$
$$\left. +(2\omega\varepsilon_\nu - kp_\nu)\left[\frac{m_e^2}{(p_e k)^2}-\frac{\varepsilon_e+\omega}{\omega(p_e k)}\right]+\frac{\varepsilon_\nu}{\omega}-\frac{2\varepsilon_\nu\varepsilon_e - p_e p_\nu}{(p_e k)}\right\}, \qquad (6)$$

where $d\Gamma_0$ is the differential width in the Born approximation, and ε_e, ε_ν, and ω are the energies of the electron, neutrino, and photon, respectively. The phase space volume in (5) can be rearranged in the following way:

$$\int \frac{d^3k}{\omega}\frac{d^3p_e}{\varepsilon_e}\frac{d^3p_\nu}{\varepsilon_\nu}\frac{d^3p_0}{\varepsilon_2}\delta(p_1-p_0-p_e-p_\nu-k)$$
$$=\frac{1}{m}\int p_e d\varepsilon_e\varepsilon_\nu d\varepsilon_\nu k d\omega 2(2\pi)^3 dc_\nu dc_\gamma\delta(\Delta-\varepsilon_e-\varepsilon_\nu-\omega)$$
$$=16\pi^3 m_\pi^{-1}\Delta^5\int_\mu^1 dx\int_\nu^{1-x}(1-x-z)dzxz\beta\beta_z\int_{-1}^1 dc_\nu\int_1^1 dc_\gamma, \qquad (7)$$

$$\beta=\sqrt{1-\frac{\mu^2}{x^2}},\quad \beta_x=\sqrt{1-\frac{\nu^2}{z^2}},\quad z=\frac{\omega}{\Delta},\quad x=\frac{\varepsilon_e}{\Delta},\quad \nu=\frac{\lambda}{\Delta},$$

where λ is the "photon-mass" parameter, and c_ν and c_γ are the cosines of the angles between the electron and the neutrino and the photon moments, respectively. We omit in (7) the terms on the order of Δ/m_π compared with the terms on the order of unity. The corresponding error is

$$\frac{\alpha}{\pi}\frac{\Delta}{m_\pi}\sim 10^{-4}. \qquad (8)$$

Performing the angular integration over c_ν, c_γ, and z, we obtain

$$\int d\Gamma^{\pi^+ \to \pi^0 e^+ \nu\gamma} = \frac{G^2\Delta^5|V_{ud}|^2\alpha}{\pi^4}\int_\mu^1 dx\,\beta x^2(1-x)^2\left\{\left(-2+\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}\right)\right.$$

$$\times\left[-1+\ln(1-x)+\frac{1-x}{3x}+\ln\frac{\Delta}{\lambda}+\frac{(1-x)^2}{24x^2}\right]$$

$$\left.+2(1-\ln 2)+\frac{(1-x)^2}{12x^2}+\frac{1}{\beta}\int_0^\beta\frac{dt}{1-t^2}\ln\frac{\beta^2-t^2}{\beta^2}.\right\} \tag{9}$$

To calculate the correction due to a "virtual photon emission," we must consider three one-loop Feynman diagrams (Fig. 2). Two of them (Figs. 2b and 2c), which contain the "contact" vertices, have the form of a pion and a positron self-energy loop. The double interference of their amplitudes with the Born one (summed over the final spin states) has the form

$$2\sum M_0^*(M_b^V+M_c^V)=24m_\pi^2(2G|V_{ud}|)^2\left(-\frac{\alpha}{4\pi}\right)\varepsilon_e\varepsilon_\nu$$

$$\times\left[\ln\frac{\Lambda^2}{m_\pi^2}+\frac{3}{2}+\frac{m_e^2}{\varepsilon_e m_\pi}\left(\ln\frac{\Lambda^2}{m_e^2}+\frac{3}{2}\right)\right]. \tag{10}$$

The contributions from the terms in this equation, which are proportional to m_e/m_π, are on the order of (8) and we will omit them below. The parameter Λ here is the momentum cutoff. Deriving (10), we omit the terms of order

$$\frac{\alpha}{\pi}\frac{m_\pi^2}{\Lambda^2}<\frac{\alpha}{\pi}\frac{m_\pi^2}{m_\rho^2}<10^{-4}. \tag{11}$$

These terms present the real magnitude of the uncertainties of the strong interaction. Applying the standard procedure for joining the denominators and performing a loop momentum integration, we obtain an expression, which is analogous to (10), for the first diagram

$$2\sum M_0^*M_a^V=16m_\pi^2(2G|V_{ud}|)^2\left(-\frac{\alpha}{4\pi}\right)\varepsilon_e\varepsilon_\nu$$

$$\times\left[-\frac{7}{2}\ln\frac{\Lambda^2}{m_\pi^2}-\frac{25}{6}+2\left(\ln\frac{m_\pi^2}{m_e^2}-\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta}\right)\right.$$

$$\left.+\frac{2}{\beta}\ln\frac{1+\beta}{1-\beta}\left(\ln\frac{m_\pi^2}{\lambda^2}+\ln\frac{\beta^2\Delta^2x^2}{m_\pi^2}\right)+\frac{4}{\beta}\int_0^\beta\frac{dt}{1-t^2}\ln\frac{1-t^2}{t^2}\right]. \tag{12}$$

To consider the ultraviolet behavior of the amplitudes correctly, it is necessary to take into account the renormalization constants Z_e and Z_π for the positron and pion wave functions:

$$\frac{1}{\hat{p}-m}\to\frac{Z_e}{\hat{p}-m},\quad\frac{1}{p_1^2-m_\pi^2}\to\frac{Z_\pi}{p_1^2-m_\pi^2}, \tag{13}$$

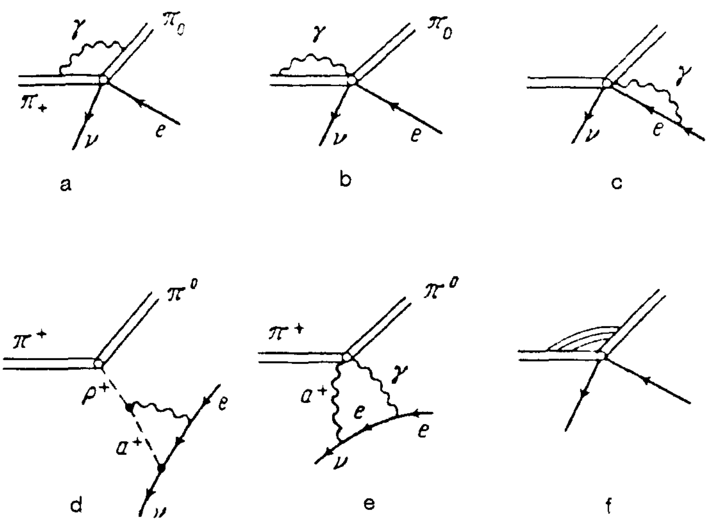

FIG. 2. One-loop Feynman diagrams for the $\pi\beta$ decay.

where

$$Z_e = 1 + \frac{\alpha}{2\pi}\left(\ln\frac{m_e^2}{\lambda^2} - \frac{1}{2}\ln\frac{\Lambda}{m_e^2} - \frac{9}{4}\right),$$

$$Z_\pi = 1 + \frac{\alpha}{2\pi}\left(\ln\frac{\Lambda^2}{m_\pi^2} + \ln\frac{m_\pi^2}{\lambda^2} - \frac{3}{4}\right). \tag{14}$$

The total sum of the radiative $\pi^+\beta$ decay width, which includes the loop corrections, does not contain the "photon mass" parameter λ:

$$\Gamma_0 + \Gamma^{(1)} = \Gamma_0(1+\delta_\pi),$$

$$\Gamma_0 = (G|V_{ud}|)^2\left(1 - \frac{3\Delta}{2m_\pi}\right)\frac{\Delta^5}{\pi^3} I(\mu), \tag{15}$$

$$\delta_\pi = \frac{\alpha}{\pi}\left(\int_\mu^1 dx x^2(1-x)^2\beta\right)^{-1}\int_\mu^1 dx x^2(1-x)^2\beta$$

$$\times\left\{\frac{1}{\beta}\int_0^\beta \frac{dt}{1-t^2}\ln\frac{(\beta^2-t^2)t^2}{1-t^2} + \frac{1}{\beta}\left(-\frac{1}{2} + \ln\frac{(1-x)}{\beta^2 x}\right.\right.$$

$$\left.+\frac{(1-x)(1+7x)}{24x^2}\right)\ln\frac{1+\beta}{1-\beta}$$

$$\left.-\frac{2(1-x)}{3x} + \frac{143}{48} - 2\ln 2 - 2\ln(1-x) + \ln\mu^2 - \frac{1}{4}\ln\frac{m_\pi^2}{m_e^2} + \frac{3}{4}\ln\frac{\Lambda^2}{m_\pi^2}\right\}. \tag{16}$$

Numerical calculations give $\delta_\pi = -0.015$ for $\Lambda = m_p$.

2. Contribution of diagrams in Figs. 2d–2f and the background process

In Ref. 7 it was argued that the pion form factor cannot remove all UV divergences. It was shown that the diagrams in Figs 2d and 2e have them. The diagram in Fig. 2d contains a vertex with a meson, a photon, and a ρ-meson. Such vertices are absent in a realistic chiral Lagrangian theory with vector current conservation. The Wess–Zumino–Witten anomalous Lagrangian L_{WZW} was obtained by calibration on the SU(3)·SU(3) group with the residual anomaly in the W. Bardeen form[8] does not contain a vertex without pseudoscalar mesons. As for the contribution of the diagram in Fig. 2e, it is described by the following term:

$$L_{WZW}=\varepsilon^{\mu\nu\lambda\sigma}(d_\mu\pi^+)(d_\nu\pi^0)e^\lambda(a)e^\sigma(\gamma).$$

We see that the contribution of this diagram to δ_π is UV finite. It has the order

$$\frac{\alpha}{\pi}\frac{m_+m_0}{m_+}\leqslant 10^{-4}, \tag{17}$$

which is less than the required accuracy. The diagrams of the type in Fig. 2f contain the form factor F_π which is induced by strong interactions of the initial and final pions. It deviates from its static limit (unity) by terms quadratic in the pion mass difference, $F_\pi\sim 1+O[((m_+-m_0)/m_+)^2]$.

As a possible background process to the pion beta decay we can consider the π_{e2} decay with two additional emitted photons:

$$\pi^+\rightarrow e^+ + \nu_e + \gamma + \gamma.$$

This process can imitate the final state of the pion beta decay with a subsequent decay of π_O into two photons. The contribution of the background process was considered in detail in Ref. 9. For a reasonable deviation δm^2 of the squared two-photon invariant mass m^2 from m_π^2 the corresponding contribution to δ_π would be small:

$$\left(\frac{\alpha m_e}{\Delta}\right)^2\left(\ln\frac{\Delta}{m_e}\right)^2\frac{\delta m^2}{m^2}\sim 10^{-4}\frac{\delta m^2}{m^2}. \tag{18}$$

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