Structure of drift surfaces and condition for orthogonality of the magnetic-field geometry

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The splitting and the nonomnigenous nature of the drift surfaces of charged particles in mirror magnetic confinement systems are shown to result from a disruption of the mutual orthogonality of the magnetic lines of force and the lines of B = const on magnetic surfaces. A condition for orthogonality of the field geometry is derived.

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The splitting[1–3] and the nonomnigenous nature[4,5] of the charged-particle drift surfaces in axially asymmetric confinement systems with magnetic mirrors determine the transverse transport rate of the plasma[6,7] and the conditions for MHD equilibrium.[5] These aspects of the drift surfaces can be analyzed from a common standpoint by associating these effects with certain metric characteristics of a natural coordinate system which is determined by the lines of force and by the distribution of the modulus of the field B in the plasma confinement region.

We will restrict the present paper to the case in which there is a single minimum of B on each line of force which penetrates into the confinement region. We consider the surface of minimum $BB_s'(\mathbf{r}) = 0$ (the subscript "s" denotes differentiation along the field direction). The family of magnetic lines of force that cross the closed curve $B = B_0$ = const, which lies in the surface $B_s'(\mathbf{r}) = 0$, forms a magnetic surface which we adopt as the coordinate surface ξ^1 = const. We adopt the intersections of

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the magnetic surface with the surfaces of constant mirror ratio $R(\mathbf{r}) = B(\mathbf{r})/B_0 = \text{const}$ as the coordinate lines ξ^2. Obviously, we have $\xi^2 B = \text{const}$ on the lines. As the coordinate lines ξ^3 we adopt the field lines on the magnetic surfaces $\xi^1 = \text{const}$.

The Larmor center of a particle with a magnetic moment $\mu = m\text{v}_\perp^2/2B$ and with a zero longitudinal velocity $\text{v}_{||} = \mathbf{vB}/B = 0$, which is at the field minimum $B_{\min} = B_0$ on the magnetic surface $\xi^1 = \text{const}$, does not leave this surface; it drifts along the line of minimum $B = B_0 = \text{const}$. This feature of the drift of a particle with $\text{v}_{||} = 0$ is an obvious consequence of the conservation of the total energy, $m\text{v}^2/2$, and of the magnetic moment μ. If v_{dN}, the drift velocity normal to the magnetic surface $\xi^1 = \text{const}$, is equal to zero everywhere in the particle confinement region, then the Larmor center drifts along the surface $\xi^1 = \text{const}$ and does not leave the surface even if $\text{v}_{||} \neq 0$. In other words, the drift surfaces do not split, and they are omnigenous. In the opposite case, the Larmor center of a particle with $\text{v}_{||} \neq 0$ leaves the magnetic surface. If $\text{v}_{||} \neq 0$, but if the displacements of the Larmor center normal to the magnetic surface cancel out exactly over one period of the motion between the bounce points (over one bounce period), then the drift surface turns out to be split, but its average position again coincides with the magnetic surface, so that the drift surfaces remain omnigenous. If the normal drift does not cancel out over a bounce period, then a displacement of the Larmor center with respect to the magnetic surface builds up and leads to an intersection of drift surfaces with different pitch angles. In other words, the drift surfaces are not omnigenous.

It follows immediately from the drift equation in the form[8]

$$\text{v}_d = \mathbf{b}[\text{v}_{||} - (\text{v}_{||}^2/\omega_c)(\mathbf{b}\cdot\text{rot}\,\mathbf{b})] + (\text{v}_{||}/\omega_c)\text{rot}\,\text{v}_{||}\mathbf{b}$$

that the drift velocity normal to the magnetic surface, v_{dN}, may be written

$$\text{v}_{dN} = (\text{v}_{||}/\omega_c)\mathbf{N}\cdot\text{rot}\,\text{v}_{||}\mathbf{b}, \qquad (1)$$

where $\mathbf{N}$ is the initial outward normal to the magnetic surface, ω_c is the cyclotron frequency, and $\mathbf{b} = \mathbf{B}/B$. If $\text{rot}\,\mathbf{B} = 0$ and $\nabla\phi$, where ϕ is the electrostatic potential, we find, after some straightforward calculations,

$$\text{v}_{dN} = (1/m\,\omega_c B)(\mu B + m\,\text{v}_{||}^2)B_s'\,\text{ctg}\,\theta, \qquad (2)$$

where θ is the angle between the magnetic line of force and the line $B = \text{const}$ $(0 < \theta < \pi)$ on the magnetic surface. We see from this expression that we have $\text{v}_{dN} = 0$ if the coordinate lines ξ^2 and ξ^3, which are formed by the lines $B = \text{const}$ and the field lines, respectively, on the $\xi^1 = \text{const}$ magnetic surface, are mutually orthogonal, i.e., if $\theta = \pi/2$. We will label a magnetic configuration having this property as a "magnetic configuration with an orthogonal field geometry." It follows from our discussion that the set of configurations with an orthogonal field geometry constitutes part of the set of configurations which have omnigenous drift surfaces in which $\theta \neq \pi/2$ and, correspondingly, $\text{v}_{dN} \neq 0$, but for which the displacement Δ_N of the Larmor center with respect to the magnetic surface $\xi^1 = \text{const}$, averaged over the bounce period, vanishes, i.e.,

$$\Delta_N = (1/T_b)\int_0^{T_b} \text{v}_{dN}\,dt = 0. \qquad (3)$$

For example, a simple mirror system with an auxiliary axial conductor which causes the lines of force to spiral in a helix, or the magnetic field of a tokamak, does not have the property of orthogonality, while it does have omnigenous drift surfaces.

To find the orthogonality condition, we write the contravariant component of $\operatorname{rot}\mathbf{B}=0$:

$$\operatorname{rot}^1\mathbf{B} = (1/\sqrt{g})[(\partial B_3/\partial\xi^2) - (\partial B_2/\partial\xi^3)] = 0, \tag{4}$$

where g is the determinant of the metric tensor g_{ik}. Using $\cos\theta = g_{23}/\sqrt{g_{22}g_{33}}$, $B_2 = (g_{23}/\sqrt{g_{33}})B$, $B_3 = \sqrt{g_{33}})B$, $\partial B/\partial\xi^2 = 0$ and $\operatorname{div}\mathbf{B} = [(1/\sqrt{g})\partial(B\sqrt{g}/\sqrt{g_{33}})/\partial\xi^3] = 0$, we find

$$\partial\sqrt{g_{33}}/\partial\xi^2 = (\sqrt{g}/\sqrt{g_{33}})(\partial/\partial\xi^3)(\sqrt{g_{22}g_{33}}\cos\theta/\sqrt{g}). \tag{5}$$

If

$$g_{33} = g_{33}(\xi^1, \xi^3), \tag{6}$$

i.e., if the component g_{33} is constant on the lines of ξ^2 or, equivalently, on the lines of $B = \text{const}$, then we have $\partial\sqrt{g_{33}}/\partial\xi^2 = 0$ and, correspondingly,

$$\cos\theta = C(\xi^1, \xi^2)\sqrt{g}/\sqrt{g_{22}g_{33}}. \tag{7}$$

It follows from (7) that if we have $\cos\theta = 0$ anywhere on a line of force then under condition (6) we have $\cos\theta = 0$ on the entire line. Condition (6) is equivalent to

$$B_s' = B_s'(\xi^1, \xi^3). \tag{8}$$

Here $B_s' = \mathbf{b}\cdot\nabla B = b^i\partial B/\partial\xi^i = b^3\,\partial B/\partial\xi^3 = (1/\sqrt{g_{33}})(\partial B/\partial\xi^3)$, since $b^1 = b^2 = 0$ and $b^3 = 1/\sqrt{g_{33}}$. On the lines of ξ^2, however, we have $B = \text{const}$, so that condition (8) also holds if (6) does. Let us consider the unit vector $\mathbf{t}$ tangent to the line $B = \text{const}$ on a magnetic surface. Obviously, we have $\mathbf{t}\cdot\nabla B = 0$ and, by virtue of (8), $\mathbf{t}\cdot\nabla B_s' = 0$. Consequently, we can write $\mathbf{t} = \lambda B \times B_s'$, where λ is some scalar. If $\cos\theta = \mathbf{t}\cdot\mathbf{b} = 0$, we find, by substituting $\mathbf{t}$,

$$(\nabla B \times \nabla B_s')\cdot\mathbf{b} = 0, \tag{9}$$

which is a necessary condition for orthogonality of the field geometry. Using the identity $\nabla B \equiv \mathbf{k}B + \mathbf{b}(\mathbf{b}\cdot\nabla B)$ ($\operatorname{rot}\mathbf{B} = 0$), where $\mathbf{k}$ is the curvature of the line of force, we can rewrite condition (9) as

$$B_{s\tau}'' = 0, \tag{10}$$

where the mixed derivative is evaluated along the directions of the line of force and of the binormal to it.

It can be shown that orthogonality condition (9) or (10) is also a sufficient condition.

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