Two-dimensional scalar equilibrium equation for a stellarator

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A two-dimensional scalar equilibrium equation is derived for plasmas in a stellarator. This equation, which is analogous to the familiar equation for a tokamak, can be used to calculate the maximum plasma pressure.

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The equilibrium, stability and evolution of a plasma of sufficiently high pressure in axisymmetric magnetic confinement systems such as a tokamak can be calculated by using a two-dimensional nonlinear scalar equation for the poloidal flow ψ. This equation can be written as follows:

$$r^2 \operatorname{div} \frac{\nabla \psi}{r^2} = -4\pi^2 r^2 \frac{dp(\psi)}{d\psi} - F(\psi)\frac{dF(\psi)}{d\psi}, \qquad (1)$$

where r is the distance to the major axis of the plasma torus, $p(\psi)$ is the plasma pressure, and $F(\psi)$ is the poloidal current flowing through the loop (r = const). The magnetic field is determined by the equation $\mathbf{B} = ([\nabla\psi\nabla\zeta] + F\nabla\zeta)/2\pi$, where ζ is the polar angle of the cylindrical coordinate system r, ζ, and z. For numerical calculations, it is convenient to write this equation in the flux coordinate system $x^1 \equiv a$, $x^2 \equiv \theta$, and $x^3 \equiv \zeta$ which is associated with the internal geometry of the system. In this case $\psi = \psi(a)$, and the sought-for functions are $r = r(a, \theta)$ and $z = z(a, \theta)$ (the method of inverted variables[1-3]). The square of the length in these coordinates is $dl^2 = g_{ik}dx^i dx^k$: $g_{11} = r_a^2 + z_a^2$, $g_{12} = r_a r_\theta + z_a z_\theta$, $g_{22} = r_\theta^2 + z_\theta^2$, $g_{13} = g_{23} = 0$, $g_{33} = r^2$; $g^{11} = r^2 g_{22}/g \cdot g^{12} = -r^2 g_{12}/g$, $g = r^2(r_a z_\theta - r_\theta z_a)^2$. The equilibrium equation (1) in the flux coordinate system is given by

$$\frac{\partial}{\partial a}(a_{22}\psi') - \psi'\frac{\partial a_{12}}{\partial\theta} = -4\pi^2\sqrt{g}\frac{p'}{\psi'} - \frac{FF'}{\psi'}\frac{\sqrt{g}}{g_{33}}. \qquad (2)$$

The prime denotes the derivative of a, and $a_{ik} \equiv g_{ik}/\sqrt{g}$. The θ coordinate can be found from the condition $g_{12} = 0$ (orthogonal coordinates) or by assuming that $\sqrt{g}/r^2 = \mathscr{F}(a)$ (coordinates with straight lines of force), or as a polar angle: $r = R - \rho(a, \theta)\cos\theta$, $z = \rho(a, \theta)\sin\theta$, and so on. In the third case Eq. (2) reduces to a nonlinear elliptic equation for the $\rho(a, \theta)$ function.

In this letter our goal is to derive an equation such as that in (2) for a multiperiod stellarator with a circular geometric axis.

For simplicity, we shall ignore the toroidal corrections for the vacuum stellarator magnetic field $\mathbf{B}_{st} = \nabla\Phi_{st}$,

$$\Phi_{st} = B_0 \left\{ R\zeta + \sum_l \epsilon_l \frac{Rl}{m_l} I_l\left(\frac{m_l \rho}{R}\right) \sin (l\omega - m_l \zeta) \right\} \tag{3}$$

and use the conventional "stellarator" approximation, in which the cross sections of the vacuum magnetic surfaces are close to the circular cross sections, and the parameters a/R and $\beta = 2\,\overline{p}/\mathbf{B}^2$ are small. We introduce the flux coordinates a_v, θ_v, and ζ with straight lines of force for the vacuum system of the magnetic surfaces and the X and Y coordinates which are formally associated with them:

$$X = R - a_v \cos\theta_v, \quad Y = a_v \sin\theta_v . \tag{4}$$

These coordinates are related to the cylindrical r and z coordinates by the relations[4]

$$\begin{aligned} r &= R - a_v (1+\delta)\cos\theta_v + \lambda a_v \sin\theta_v, \\ z &= a_v (1+\delta)\sin\theta_v + \lambda a_v \cos\theta_v, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \delta(a_v, \theta_v, \zeta) &= \sum_l \frac{1}{a_v} \frac{df_l}{da_v} \cos(l\theta_v - m_l \zeta), \\ \lambda(a_v, \theta_v, \zeta) &= -\sum_l \frac{l f_l}{a_v^2} \sin(l\theta_v - m_l\zeta), \end{aligned} \tag{6}$$

$$f_l = \epsilon_l \frac{R^2 l}{m_l^2} I_l\left(\frac{m_l a_v}{R}\right) . \tag{7}$$

We now introduce the flux coordinate system a, θ, and ζ in the presence of a plasma. Thus X and Y coordinates and a_v and θ_v coordinates, which are assumed to be the functions of these variables, turn out to be independent of ζ in the stellarator approximation.[5] This means that the equilibrium problem reduces to a two-dimensional problem. We shall isolate in $\alpha_{ik} = g_{ik}/\sqrt{g}$ the average part in ζ: $\alpha_{ik} = \alpha^0_{ik}(a, \theta) + \alpha_{ik}(a, \theta\zeta)$. Thus, in the approximation under consideration we have

$$\begin{aligned} a^0_{12} &= (X'\mathring{X} + Y'\mathring{Y})/RD, \quad a^0_{22} = (\mathring{X}^2 + \mathring{Y}^2)/RD, \\ a^0_{33} &= R[1 - k(R-X) + h_1(a_v)]/D, \\ (\sqrt{g})^0 &= R[1 - k(R-X) - h_2(a_v)]D, \\ D &= \mathring{X}Y' - X'\mathring{Y}. \end{aligned} \tag{8}$$

Here the prime denotes the derivative of a, the dot denotes the derivative of θ, and $k = 1/R$ is the curvature of the geometric axis. The h_1 and h_2 functions are determined by the equations

$$\begin{aligned} 2h_1 &= \sum_l \left[A^2 f_l^2 - B\frac{f_l f_l'}{a_v} + (C + m_l^2/R^2) f_l'^2 \right], \\ 2h_2 &= \sum_l \left[\frac{l^2}{a_v^2} A f_l^2 - B\frac{f_l f_l'}{a_v} + C f_l'^2 \right], \end{aligned} \tag{9}$$

where

$$A = m_l^2/R^2 + l^2/a_v^2, \quad B = m_l^2/R^2 + 3l^2/a_v^2, \quad C = \frac{l^2+1}{a_v^2}.$$

The sought-for two-dimensional scalar equilibrium equation, which can easily be determined from the general system of equilibrium equations in the flux coordinates,[3] can be written as follows:

$$\psi'\left[\frac{\partial}{\partial a}(a_{22}^0\psi') - \psi'\frac{\partial a_{12}^0}{\partial\theta}\right] = -4\pi^2(\sqrt{g})^0 p' - \frac{FF'}{a_{33}^0} - \frac{DF\psi'}{R^2}\frac{1}{a_v}\frac{d}{da_v} \times [a_v^2\mu_0(a_v)]. \tag{10}$$

It differs from Eq. (2) in that it has stellarator corrections in $(\sqrt{g})^0$ and in α_{33} and an additional term containing the vacuum rotational transformation μ_0

$$\mu_0(a_v) = \sum_l \epsilon_l^2 \frac{m_l l^3}{4\xi}\frac{d}{d\xi}\left[\frac{1}{\xi}\frac{dI_l^2(\xi)}{d\xi}\right], \quad \xi = \frac{m_l a_v}{R}. \tag{11}$$

After solving Eq. (10) as $X(a, \theta)$ and $Y(a, \theta)$ functions or as $a_v(a, \theta)$ and $\theta_v(a, \theta)$ functions, we can convert, according to Eq. (5), to the r, z coordinates of the laboratory coordinate system, thus taking the three-dimensional equilibrium configuration into account.

In solving Eq. (10), we took advantage of the fact that the vacuum magnetic surfaces are close to the circular surfaces. Since the distortions associated with the plasma pressure are large, Eq. (10) can be used to study stellarators in the same way as Eqs. (1) and (2) are used for tokamaks. We thus can solve the two-dimensional scalar equation by using the well-known methods of solving an analogous equation in the tokamak theory, instead of using the complex, three-dimensional calculations for the optimization of stellarators, which takes into account the limiting plasma pressure.

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