Stability of weak shock waves with a finite-width front

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The weak shock waves are shown to be stable with respect to transverse modulations.

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1. Elsewhere,[1] the author has participated in a study of the stability of the solutions of Burgers equation which describes weak shock waves with a finite width of the front, $\sim 2\mu/\mathrm{v}^2$

$$u_0(x - \mathrm{v}t) = \mathrm{v}\left[1 - \mathrm{th}\,\frac{\mathrm{v}}{2\mu}(x - \mathrm{v}t)\right] . \qquad (1)$$

The analysis was made within the framework of the equation given in Ref. 1,

$$\frac{\partial}{\partial x}(u_t + uu_x - \mu u_{xx}) = -\frac{c_s}{2}\Delta_\perp u , \qquad (2)$$

which takes the weak transverse modulation of such waves into account and is analogous to the Kadomtsev-Petviashvili equation for multidimensional waves in weakly dispersive media[3] [in Eqs. (1) and (2) μ is the coefficient of "viscosity" and c_s is the velocity of sound].

The linearization of Eq. (2) against the background of the steady-state solution of Eq. (1) leads to the following spectral problem for the third-order operator:

$$\frac{\partial}{\partial x}\left[-\frac{\partial^2}{\partial x^2} - 2\,\mathrm{th}x\frac{\partial}{\partial x} - \frac{2}{\mathrm{ch}^2 x} + p\right]\delta u = \beta\,\delta u, \qquad (3)$$

where the dimensionless variables $x \to (x - \mathrm{v}t)\mathrm{v}/2\mu$ and $t \to t\mathrm{v}^2/4\mu$ are introduced; the perturbation is chosen in the form $\delta u(x)e^{pt + i\mathbf{k}_\perp \mathbf{r}_\perp}$ and $\beta = 4c_s\mu^2 k_\perp^2/\mathrm{v}^3$. The finite conditions for the solutions of $\delta u(x)$ at infinity serve as the boundary conditions of Eq. (3). The presence or absence of an instability corresponds to the presence or absence of solutions of Eq. (3) with Re$p > 0$ (which have finite values in the limit $x \to \pm\infty$).

A perturbation theory for Eq. (3) was constructed in Ref. 1 using the small parameter β. Because of an error in the calculations (the system of functions used in the expansion was not complete), an erroneous conclusion was drawn with respect to the instability of the solution. In the present paper we obtain the exact solutions of Eq. (3) for arbitrary β, and we show that they are stable.

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2. We examine the asymptotic expression for the solutions of δu. We assume that δu behaves like $e^{-2\alpha x}$ in the limit $x \to +\infty$, and like $e^{2\tilde{\alpha} x}$ in the limit $x \to -\infty$. The indices α and $\tilde{\alpha}$ satisfy the following cubic equations:

$$\alpha^3 - \alpha^2 - \frac{p}{4}\alpha - \frac{\beta}{8} = 0, \tag{4}$$

$$\tilde{\alpha}^3 - \tilde{\alpha}^2 - \frac{p}{4}\tilde{\alpha} + \frac{\beta}{8} = 0. \tag{5}$$

We define the "good" indices as those with a positive real part, $\mathrm{Re}\alpha$ or $\mathrm{Re}\tilde{\alpha} > 0$ (the corresponding asymptotic expressions for δu are finite). As follows from Eqs. (4) and (5), the indices α and $\tilde{\alpha}$ pass through their critical (purely imaginary) values when the parameter p runs through the curves

$$a, a': \qquad \mathrm{Re}p = -4v^2, \qquad \mathrm{Im}\,p = -4v + \frac{\beta}{2v} \quad (\alpha = iv),$$

$$b, b': \qquad \mathrm{Re}p = -4v^2, \qquad \mathrm{Im}\,p = -4v - \frac{\beta}{2v} \quad (\tilde{\alpha} = iv)$$

in the (Rep, Imp) plane (see Fig. 1). These curves divide the complex p plane into four regions (labeled I–IV in the figure), and the number of "good" indices in these regions is as follows: I–$(3, \tilde{2})$, II–$(2, \tilde{2})$, III–$(2, \tilde{1})$, IV–$(1, \tilde{2})$. These notations mean, for example, that in region I all three roots of Eq. (4) have a positive real part, but Eq. (5) has only two roots with $\mathrm{Re}\tilde{\alpha} > 0$ ($\tilde{2}$–two good indices). The instability region $\mathrm{Re}p > 0$ is a part of region IV in which we number the roots of Eqs. (4) and (5) as follows:

$$\mathrm{Re}\alpha_3 < \mathrm{Re}\alpha_2 < 0 < \mathrm{Re}\alpha_1\,, \qquad \mathrm{Re}\tilde{\alpha}_1 < 0 < \mathrm{Re}\tilde{\alpha}_2 < \mathrm{Re}\tilde{\alpha}_3\,. \tag{6}$$

In Eq. (3) we change over to the argument $z = \frac{1}{2}(1 - \tanh x)$ (as $x \to \infty$, $z \to e^{-2x}$, and as $x \to -\infty$, $1 - z \to e^{2x}$) and we introduce into the discussion the solutions of Eq. (3)–

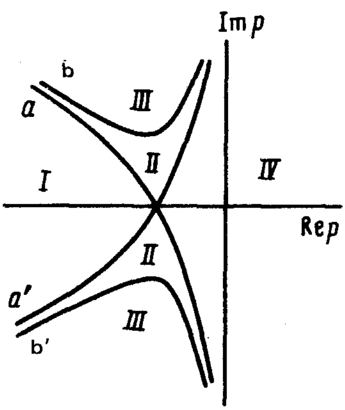

FIG. 1.

analogs of the Jost functions:

$$\Phi_i = z^{a_i}\phi_i(z), \qquad \Psi_i = (1-z)^{\tilde{a}_i}\psi_i(1-z).$$

The $\phi_i(z)$ functions are regular at zero and are normalized by the condition $\phi_i(0)=1$, and the $\psi_i(1-z)$ functions are regular at $z=1$ and are normalized by the condition $\psi_i(z=1)=1$. The solutions of Φ_i and Ψ_i are related by the elements of the scattering matrix S_p: $\Phi_i = S_{ik}\Psi_k$; specifically,

$$z^{a_1}\phi_1 = S_{11}(1-z)^{\tilde{a}_1}\psi_1 + S_{12}(1-z)^{\tilde{a}_2}\psi_2 + S_{13}(1-z)^{\tilde{a}_3}\psi_3 . \qquad (7)$$

In region IV of the complex p plane only Φ_1 of the three Φ_i functions is bounded in the limit $z\to 0$ $(x\to\infty)$, while Ψ_2 and Ψ_3 are bounded in the limit $z\to 1$ $(x\to-\infty)$. Therefore, the solution of Eq. (3), which is bounded for all x, exists only for those eigenvalues p for which $S_{11}(p)=0$. This condition defines the discrete spectrum of p (if it exists) in region IV. We note that if the solution of Φ_1 is sought in the form $(1-z)^{\tilde{\alpha}_1} f(z)$, then, as follows from Eqs. (6) and (7), $S_{11}(p)=f(1)$. We prove the absence of a discrete spectrum by a direct calculation of this quantity.

3. Substituting $\delta u = z^{\alpha_1}(1-z)^{\tilde{\alpha}_1} f(z)$ in Eq. (3), we find the following equation for f:

$$z^2(1-z)^2\frac{\partial^3 f}{\partial z^3} + z(1-z)\frac{\partial^2 f}{\partial z^2}[3a_1+2-(3\lambda+4)z] + \frac{\partial f}{\partial z}\Big\{3a_1^2 + a_1 - \frac{p}{4}$$
$$-z[(\lambda+2)(3\lambda-1)+\mu(3\lambda+1)] + z^2(\lambda+2)(3\lambda-1)\Big\}$$

$$-f\left[\frac{\mu}{2}(\lambda+1)(3\lambda-4)+\frac{1}{2}(\lambda+1)(\lambda+2)(\lambda-2) - z(\lambda+1)(\lambda+2)(\lambda-2)\right]=0,$$

where $\lambda=\alpha_1+\tilde{\alpha}_1$ and $\mu=\alpha_1-\tilde{\alpha}_1$. The expansion of the function f in a series $f=\Sigma C_n z^n/n!$ and an analysis of the recursion formula for the coefficients C_n (which we omit here) suggest an integral substitution for f of the form

$$f = \int_C dt\,(-t)^{a_1+\tilde{a}_1}(1-t)^{-a_3-\tilde{a}_1-1}\,G(zt)\,.$$

Here and below C is the contour used in the theory of hypergeometric functions. Such a substitution reduces the third-order equation for f to a second-order equation for $G(z)$:

$$z(1-z)^2\frac{\partial^2 G}{\partial z^2} + (1-z)\frac{\partial G}{\partial z}[1+a_1-a_2-z(\lambda+a_1-a_3)]$$

$$-G\left[\frac{1}{2}(\lambda+2)(\lambda-2)+\frac{\mu}{2}(3\lambda-4) - z(1+a_1-a_3)(\lambda-2)\right]=0,$$

which is reduced in turn by the substitution $G(z) = (1-z)^s F(z)$ to the standard hypergeometric equation. The exponents s, with allowance for Eqs. (4) and (5), are $s_1 = \alpha_3 + \widetilde{\alpha}_3$ and $s_2 = \alpha_3 + \widetilde{\alpha}_2$. We give an explicit expression for $f(z)$ which takes the normalization $f(0) = 1$ into account:

$$f(z) = \frac{\Gamma(1 + a_1 - a_3)\Gamma(-a_1 - \widetilde{a}_1)}{\Gamma(-a_3 - \widetilde{a}_1)(-2\pi i)} \int_C dt(-t)^{a_1 + \widetilde{a}_1}(1-t)^{-a_3 - a_1 - 1}(1 - zt)^{a_3 + \widetilde{a}_3}$$

$$\times F(-a_2 - \widetilde{a}_2,\ 1 + a_1 + \widetilde{a}_3,\ 1 + a_1 - a_2, zt). \tag{8}$$

Other solutions of the equation can be constructed in the same manner.

4. To determine the scattering coefficient $S_{11}(p) = f(1)$, we expand the hypergeometric function $F(t)$ in Eq. (8) in a series and integrate term by term. After summing the obtained series, we find the final expression

$$S_{11}(p) = \frac{\Gamma(1 + a_1 - a_3)\Gamma(1 + a_1 - a_2)\Gamma(\widetilde{a}_2 - \widetilde{a}_1)\Gamma(\widetilde{a}_3 - \widetilde{a}_1)}{\Gamma(1 + a_1 + \widetilde{a}_2)\Gamma(1 + a_1 + \widetilde{a}_3)\Gamma(-a_2 - \widetilde{a}_1)\Gamma(-a_3 - \widetilde{a}_1)}. \tag{9}$$

The zeros of S_{11} can be concentrated only at points where the arguments of the Γ functions in the denominator of Eq. (9) are equal to negative integers. Because of the conditions (6), all of these arguments have a positive real part in region IV.

Therefore, the spectrum of the eigenvalues p of Eq. (3) lies entirely outside region IV, in particular, outside the "unstable" region $\mathrm{Re}\,p > 0$. This also means that shock waves with a finite-width front are stable against transverse modulations.

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