Stability of the traveling solitons of superlattices

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The stability of self-similar solutions of the sine-Gordon equation with friction, which describe the time-varying Josephson effect, is investigated.

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The quasistationary dynamic states of systems such as a distributed Josephson contact, a variable-thickness superconducting film, interphase boundary, atomic monolayer adsorbed on a crystal surface, and others[1] can be conveniently described in soliton language: There is a sequence of equally spaced solitons, which are moving with a constant velocity v. All of these systems are described by the dissipative sine-Gordon equation[4]:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} - 2\gamma \frac{\partial \phi}{\partial t} + F - \sin \phi = 0. \tag{1}$$

In a Josephson contact $\phi(x, t)$ is the phase difference of the order parameter, and the constant F is equal to I/I_c (I_c is the Josephson critical current). Independence of x for the current I can be achieved by selecting a special contact geometry or due to the thermal effect.[2] In the quasistationary state the energy dissipated during a period T is compensated by the action of the current source, and $\phi(x, t)$ has the form

$$\phi(x, t) = \phi(\xi), \quad \xi = t + \frac{x}{\mathrm{v}}, \quad \phi(\xi + T) = \phi(\xi) + 2\pi . \tag{2}$$

Each region in which the phase changes abruptly by 2π (Fig. 1a), which is called a soliton or Josephson vortex, carries one flux quantum ϕ_0.[4] The distance l between solitons is equal to vT. The function $\phi(\xi)$ satisfies the equation

$$\left(\frac{1}{\mathrm{v}^2} - 1\right)\phi'' - 2\gamma \phi' + F - \sin \phi = 0 \tag{3}$$

whose periodic solutions have been investigated in Ref. 3. A solution for given γ, F, and v exists when $F \geqslant F_0(\gamma, \mathrm{v})$ (S and U regions in Fig. 3), and it is unique. Therefore, Eqs. (2) and (3) specify $\phi(\xi)$ and its period T if γ, F, and v are given. The shape of the soliton $\phi(\xi)$ is not the same as the normal shape of the solitons (5) of the sine-Gordon equation.

The stability of the solutions of Eq. (3) with respect to small perturbations is investigated with the aid of linearized Eq. (1),

$$-\left(\frac{1}{\mathrm{v}^2} - 1\right)\psi'' + \left(\cos \phi(\xi) - \gamma^2 - \frac{\lambda^2}{1 - \mathrm{v}^2}\right)\psi = 0, \tag{4}$$

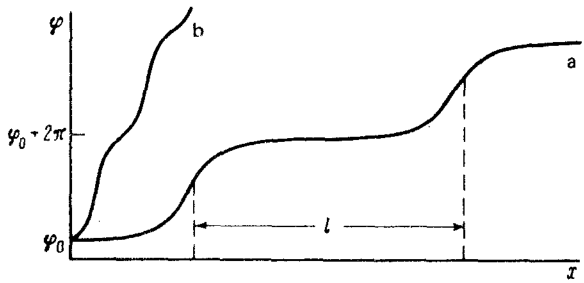

FIG. 1. a–Widely spaced solitons, b–closely spaced solitons.

which is written in the coordinate system that moves along with the solitons. It resembles the Schrödinger equation for a particle in a periodic potential $U(\xi) = \cos\phi(\xi)$; however, we must examine it for complex eigenvalues λ and quasimomenta p, which are related to the natural frequencies ω of the small oscillations and to the physical quasimomenta k by the relations $\lambda = \omega + i\gamma$ and $p = k + \lambda v^2/(1 - v^2)$, where k is real. The authors of Ref. 4 dealt only with real p; this led them to an incorrect result. The $\lambda(p)$ spectrum consists of real components (allowed bands) and some lines in the complex λ plane (Fig. 2). If even one point of the $\lambda(p)$ spectrum exists with $\mathrm{Im}\lambda > \gamma$, the corresponding solution of $\phi(\xi)$ is unstable.

Equations (3) and (4) are solved exactly in the conservative case: $\gamma = 0$ and $F = 0$. The self-similar solutions of the sine-Gordon equation, which satisfy Eq. (2), are

$$\phi = 2am\left(\frac{\xi v}{\mu\sqrt{|1 - v^2|}}, \mu\right) + \pi\theta(1 - v), \tag{5}$$

where μ is the modulus of the elliptic function. After inserting $\gamma = 0$ and $\phi(\xi)$ from Eq. (5), Eq. (4) becomes the Lamé equation. An analytic continuation of the quasimomenta p shows that $\lambda = \omega$ does not exist when the complex frequencies move slower than the speed of light ($v < 1$), and at $v > 1$ a complex bridge appears across the forbidden band (Fig. 2). When λ passes along the bridge, p bends around the cut

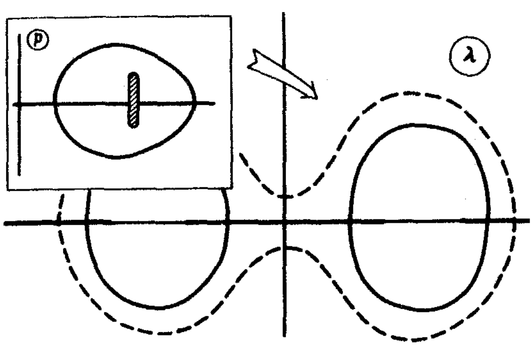

FIG. 2. Complex eigenvalues λ (spectrum) and their corresponding complex quasimomenta p (in the inset).

and intersects the real axis at the reciprocal-lattice vector π/T. A similar picture is also typical of the dissipative case. At large periods T the bridges merge into one complex component (dashed line in Fig. 2). It turns out that stability of all "slower-than-light" solutions and instability of all "faster-than-light" solutions are typical of the conservative case: This fact is demonstrated for the sine-Gordon type of equation, but with the replacement of $\sin\phi$ by an arbitrary periodic function $f(\phi)$. This instability results in the fact that only the "slower-than-light" branch is observed in experiments[5] on S-I-S contacts with very small γ and F.

For the dissipative case we were able to study a number of limiting cases and investigate the total picture qualitatively (Fig. 3). 1) All "slower-than-light" solutions of Eq. (3) are stable. 2) The "faster-than-light" solutions are stable only at $F>1$; i.e., they are observed only in the resistive state: $I>I_c$. 3) Very tightly spaced solitons (Fig. 1b) are stable for any v. This case is realized at large F, where $\phi(\xi)=(F/2\gamma)\xi-(\mathrm{v}2\gamma/\sqrt{|1-\mathrm{v}^2|}F)^2\sin(2\pi/T)\xi$, the period $T=4\pi\gamma/F$ is small, and the voltage across the contact $\partial\phi/\partial t$ contains only the first harmonic (the Josephson frequency). 4) The cutoff force $F_0(\gamma,\mathrm{v})$ corresponds to single-soliton solutions, and the vertical segment in Fig. 3: $\eta_c/\sqrt{\eta_c^2-4\gamma^2}<\mathrm{v}<\eta_c/\sqrt{\eta_c^2+4\gamma^2}$ corresponds to solitons with a power-law behavior at infinity: $\phi(\xi)=\pi/2-4\gamma/\xi+2n\pi$. These solitons are stable. The critical viscosity η_c was calculated in Ref. 6: $\eta_c\approx 1.193$. Very widely spaced and isolated solitons, which behave exponentially at infinity and which move with velocities $\mathrm{v}<1$, are stable, while those with $\mathrm{v}>1$ are not stable. In the region near the $F_0(\gamma,\mathrm{v})$ curve in Fig. 3 the periods T and l are large, and the voltage differs drastically from the sinusoidal shape. In the case of a weak dissipation ($\gamma\to 0$) the N region is narrow, and solutions with $\mathrm{v}<1$ and $F>\mathrm{v}/\sqrt{\mathrm{v}^2-1}$ (S region) are stable. As γ is increased, the instability region U shrinks at the expense of regions N and S. For $\gamma\geqslant\eta_c/2\approx 0.6$ the instability region U dies out, and the line $F\equiv 1$ becomes the boundary $F_0(\gamma,\mathrm{v})$ between N and S. We note that the time-varying conditions (2) and (3) determine the phase $\phi(\xi)$ for given γ, F, and v. However, the soliton velocity v for given γ and $F=I/I_c$ is determined by the external magnetic field and by the specific contact geometry; however, v must always lie in the S region.

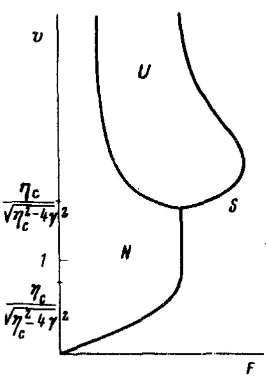

FIG. 3.

The case $v = \infty$ corresponds to states in which the phase ϕ depends only on the time ($\xi = t$). The stability of the solutions of $\phi(t)$ in Eq. (3) (with $1/v^2 = 0$) is determined from the Cauchy problem:

$$y'' + 2\gamma y' + (\cos\phi(t) + k^2)y = 0 \tag{6}$$

[the small perturbation $\phi_1(k, t) = y(t)\exp(ikx)$]. If among the solutions of $y(t)$ for a real k there is a solution that increases as $t \to +\infty$, then $\phi(t)$ is unstable. The stability limit is determined analytically for the cases $\gamma \to 0$ and $\gamma \to \infty$, but for an arbitrary γ it is determined by solving Eq. (6) on a computer. It turns out that the solutions $\phi(t)$ for $F > 1$ are stable, regardless of the value of γ, while for $F < 1$ they are not stable. Since $F = I/I_c$ and $2\pi/T = \langle \partial\phi/\partial t \rangle$ is proportional to the voltage, the volt-ampere characteristic of the in-phase oscillations in the absence of an external magnetic field is determined by the $T(F)$ dependence that follows from Eq. (3). In the case $\gamma < 0.6$ its solutions exist even for $I < I_c$, but since they are unstable, they cannot be observed experimentally, and a voltage surge occurs at $I = I_c$. The surge decreases with increasing γ and at $\gamma \geqslant 0.6$ it vanishes, while the volt-ampere characteristic acquires the shape characteristic of a point contact $\langle \partial\phi/\partial t \rangle \sim \sqrt{I - I_c}$. The in-phase oscillations in contacts of arbitrary length have the same properties.

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