Pulse propagation in a long laser amplifier

S. V. Manakov
L. D. Landau Institute of Theoretical Physics

(Submitted 22 December 1981)
Pis'ma Zh. Eksp. Teor. Fiz. **35**, No. 5, 193–195 (5 March 1982)

The characteristic of a long laser amplifier is determined; i.e., the structure of a pulse at the amplifier output is determined as a function of the shape of the ignition pulse.

PACS numbers: 42.60.Da

1. The propagation of a coherent pulse in a two-level nondegenerate medium, if the dissipative effects are ignored, is described by the known system of equations[1,2]

$$E_x + E_t = 2\pi i \Omega\, d \int n(\omega) u(\omega) \bar{v}(\omega)\, d(\omega), \tag{1}$$

$$\begin{aligned} u_t &= i\omega u + i d E v, \\ v_t &= -i\omega v + i d \bar{E} u \end{aligned} \tag{2}$$

for the complex envelope of an electric field E and for the amplitudes of the probabilities u and v of a "two-level atom" residing in the upper (or lower) states. Here Ω and d are the frequency and dipole moment of the transition, and the function $n(\omega) = N_+(\omega) - N_-(\omega)$, which characterizes the inhomogeneous broadening, is the difference between the initial populations of the upper and lower levels. We assume that the medium occupies the half-space $x > 0$ and that there is no field in the medium at $t \leqslant 0$, and at the instant of time $t = 0$ the pulse $E(x, t)|_{x=0} = E_0(t)$, where $E_0(t) = 0$ at $t < 0$, enters the medium. The notations in Eqs. (1) and (2) were chosen in such a way that we must assume $u(\omega, x, t) = 1$ and $v(\omega, x, t) = 0$ at $t = 0$. We describe $E(x, t)$ as a function of $E_0(t)$ at sufficiently large x in the case of an inversion-population medium $N = \int n(\omega) d\omega > 0$; i.e., we describe the "characteristic" of a long amplifier.

2. The system of Eqs. (1) and (2) can be investigated in detail by using the inverse-problem method. Without dwelling on details, we present here only the needed facts; the details can be found elsewhere.[3,4] It is not difficult to relate the notations used here with those in Ref. 3.

We use $\chi(\omega, x, t)$ to denote the vector

$$\chi = (\chi_1, \chi_2) = (u(\omega, x, t), v(\omega, x, t))\, e^{-i\omega t}.$$

The function χ satisfies the equation

$$\chi(\omega) = (1, 0) - \frac{1}{2\pi i} \int_C R(\omega', x) e^{-2i\omega' t}\, \tilde{\chi}(\omega') \frac{d\omega'}{\omega' - \omega}, \tag{3}$$

where $\tilde{\chi} = (-\bar{\chi}_2, \bar{\chi}_1)$, and $R(\omega', x)$ is the conventionally defined reflection coefficient

 0021-3640/82/05237-04$01.00

for the system (2). The dependence of R on x is given by

$$R(\omega, x) = R(\omega,\ 0)\exp 2ix\left(\omega - \frac{1}{2}\int \frac{\pi\,\Omega\, d^2 n(\omega')}{\omega' - \omega - i\,0}\, d\omega'\right) . \tag{4}$$

$R(\omega, 0)$ is calculated from the incoming pulse $E_0(t)$. The integration contour C in Eq. (3) extends into the upper half-plane.

Equations (3) and (4) are the system of equations of the inverse problem, whose solutions determine $u(\omega, x, t)$ and $v(\omega, x, t)$ and, therefore, $E(x, t)$.

We shall assume that $E_0(t)$ has a power-law behavior at zero, where $E_0(t) = ct^{\nu}$ at $t>0$. Without loss of generality, we can assume that c is real. For such E_0 the asymptotic behavior of $R(\omega, 0)$ in the limit $\omega \to \infty$ and at $\mathrm{Im}\,\omega > 0$ has the form

$$R \simeq c'\,\omega^{-(\nu+1)}, \qquad c' = -i^{\nu} 2^{-(\nu+1)}\Gamma(\nu + 1)\, dc \tag{5}$$

[we show below that the entire structure of $E(x, t)$ at large x is determined by the behavior of E_0 at zero and is insensitive to the subsequent behavior of this function. In other words, the pulse shape in the long amplifier is determined exclusively by the ignition-pulse front].

3. We introduce the notations

$$z = 4\,\Omega_0\,\sqrt{x(t-x)}, \qquad \Omega_0^2 = \frac{\Omega\,\pi\, d^2}{2}\int n(\omega)\, d\omega .$$

At large x and intermediate z ($z \ll \Omega_0 x$) Eqs. (3) [with allowance for Eqs. (4) and (5)] are simplified to the form

$$\chi(\Lambda) = (1,\ 0) - \frac{f_\nu}{2\pi i}\int_C \frac{e^{-i\frac{z}{2}(\Lambda' - \frac{1}{\Lambda'})}}{\Lambda' - \Lambda}\, \frac{\tilde{\chi}(\Lambda')}{\Lambda'^{\,\nu+1}}\, d\Lambda', \tag{6}$$

where

$$\Lambda = \frac{z}{4\Omega_0^2 x}\,\omega, \qquad f_\nu = c'\left(\frac{z}{\Omega_0^2 x}\right)^{\nu+1} . \tag{7}$$

At $z \gg 1$ the integral in Eq. (6) is determined by the neighborhood of the saddle point $\Lambda' = i$. A formal computation of the right side of Eq. (6) with an accuracy to $O(e^z/z^{N+1/2})$ gives an expression for $\chi(\Lambda)$ in terms of the function $\tilde{\chi}(\Lambda)$ and its $2N$ first derivatives at $\Lambda = i$. This makes it possible to write a closed system of linear algebraic equations for $\chi, \chi', \ldots, \chi^{(2N)}|_{\Lambda=-i}$ and $\tilde{\chi}, \tilde{\chi}', \ldots, \tilde{\chi}^{(2N)}|_{\Lambda=i}$, whose solution determines $\chi(\Lambda)$ and, ultimately, $E(x, t)$. The answer looks especially simple in the limit of very large x: if $\ln\ln \Omega_0 x > 1$, then E near the pulse front has the form of a sequence of 2π pulses

$$E(x, t) = 8\,\Omega_0^2\, x \sum_{k=1}^{N+1} \frac{(-1)^k}{z_k}\,\mathrm{ch}^{-1}\left[\frac{8\Omega_0^2\, x}{z_k}\,(t - x - \xi_k)\right], \tag{8}$$

where $z_k \simeq (\nu+1)\ln \Omega_0 x - (\nu + \frac{3}{2} - k)\ln\ln \Omega_0 x$ and $\xi_k = (1/x)(z_k/4\Omega_0)^2$. In this limit the pulse front therefore consists of a sequence of alternating 2π and -2π pulses of decreasing width $\tau \simeq (\nu+1)(\ln \Omega_0 x)/(8\Omega_0^2 x)$ and increasing amplitude $|E| \simeq 8\Omega_0^2 x(\nu+1)\ln \Omega_0 x$. The distance between the solitons behaves as $\Delta\tau \approx 2\tau \ln\ln \Omega_0 x$. For a given N, Eq. (8) applies at $z < z_{N+1} + \frac{1}{2}\ln\ln \Omega_0 x$. We note, however, that this answer is generally valid only at $z < z_{N_{\max}}$, where $N_{\max} \sim \ln \Omega_0 x/\ln\ln \Omega_0 x$.

4. It follows from Eqs. (6) and (7) that the quantity $U = d\int_0^t E(x, t')dt'$ satisfies the equation

$$U_{xt} + U_{tt} = 4\Omega_0^2 \sin U. \tag{9}$$

This equation has self-similar solutions (see Refs. 5 and 6) that depend only on $z = 4\Omega_0\sqrt{x(t-x)}$, such that

$$U_{zz} + \frac{1}{z}U_z = \sin U. \tag{10}$$

The solutions of this equation, which are regular at zero, are uniquely determined by the solution at zero

$$U = U(U_0, z), \qquad U(U_0, 0) = U_0,$$

so that U_0 parametrizes the set of solutions (10).

It is important to note that the parameter U_0 can be described by a sufficiently arbitrary, slow function of x/z. In this case $U[U_0(x/z), z]$ is everywhere only slightly different from the true solution of Eq. (9). The fact that such "quasi-self-similar" solutions arise in this problem is seen directly from Eqs. (6) and (7): The dependence of χ on the coordinates x and t converges, for the most part, to a z dependence, since the solution, roughly speaking, depends on f_ν only logarithmically.

The specific dependence on x/z can be determined in the following manner. For $U_0 \ll 1$ the solution of (10) has the form

$$U = U_0 I_0(z), \tag{11}$$

where I_0 is the Bessel function of an imaginary argument. If $\ln 1/U_0 > 1$, Eq. (11) is valid at large z when I_0 can be replaced by its asymptotic form

$$U = U_0 e^z/\sqrt{2\pi z}.$$

The corresponding expression for E has the form

$$E = \frac{8U_0}{d\sqrt{2\pi}}\Omega_0^2 \frac{x}{z^{3/2}} e^z. \tag{12}$$

On the other hand, at large x and z the solution of Eq. (6) can be obtained by a simple iteration in the region where the inequality $f_\nu e^z < 1$ is satisfied. In this region

$$Ed = -8e^{i\frac{\pi \nu}{2}} \bar{f}_\nu \frac{\Omega_0^2 x}{z} I_\nu (z) \simeq -\frac{8}{\sqrt{2\pi}} i^\nu \bar{f}_\nu \left(\frac{z}{x}\right) \frac{\Omega_0^2 x}{z^{3/2}} e^z . \tag{13}$$

Comparing Eqs. (12) and (13), we find

$$U_0 = c\Gamma(\nu + 1)(z/8\,\Omega_0^2\, x)^{\nu+1} . \tag{14}$$

The expression for the field $E = d^{-1} U_t$ has the form

$$E(x, t) = \frac{8\Omega_0^2 x}{dz} U_z\,(U_0(z/x), z). \tag{15}$$

The region of applicability of Eqs. (14) and (15) is given by the inequalities $z \gg 1$ and $\ln 1/U_0 \gg 1$.

Thus, the obtained expression for the field in a long laser amplifier has a quasi-self-similar character. The field in the pulse increases approximately as x, and the pulse width decreases as $1/x$. The structure of the pulse is determined exclusively by the front of the ignition pulse [the parameters c and ν in Eq. (14)].

As regards the explicitness of Eq. (15), although the solution of Eq. (10) is generally not expressed in terms of tabulated functions (they belong to the so-called Penlev transcendental functions), $U(U_0, z)$ is approximated in the limit $\ln 1/U_0 > 1$ by elliptic functions with slowly varying parameters.

The author wishes to thank V. E. Zakharov for attracting his attention to this group of problems, and I. Gabitov and A. Mikhaĭlov for fruitful discussions.

1. S. L. McCall and E. L. Hahn, Phys. Rev. 103, 183 (1969).
2. G. L. Lamb, Jr., Rev. Mod. Phys. 43, 99 (1971).
3. M. J. Ablowitz, D. I. Kaup, and A. C. Newell, J. Math. Phys. 15, 1852 (1974).
4. V. E. Zakharov, S. V. Manakov, S. P. Novikov, and I. P. Pitaevskii, Teoriya solitonov: Metod obratnoĭ zadachi (Theory of Solitons: The Inverse-Problem Method), Nauka, Moscow, 1980.
5. G. L. Lamb, Jr., Phys. Lett. 29A, 507 (1969).
6. V. E. Zakharov, Pis'ms Zh. Eksp. Teor. Fiz. 32, 603 (1980) [JETP Lett. 32, 589 (1980)].

Translated by Eugene R. Heath
Edited by S. J. Amoretty