Angular condition imposed on the state vector of a compound system for a light front

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(Submitted 12 January 1982)
Pis'ma Zh. Eksp. Teor. Fiz. **35**, No. 5, 220–223 (5 March 1982)

An equation arising from an analysis of the four-dimensional rotations of a light-front hypersurface is derived for the state vector defined for a light front $\omega x = \sigma$ ($\omega^2 = 0$). This equation supplements the Schrödinger equation and removes the ambiguities in the wave functions of relativistic compound systems.

PACS numbers: 03.65.Ca

A relativistic wave-function (WF) system, which is needed to describe compound systems (nuclei and hadrons in quark models) at momenta of the order of their masses, has been developed.[1–5] These WF are the Fock components of the state vector defined for the light-front surface $\omega x = 0$ ($\omega^2 = 0$). The convenience of such a tool, as compared, for example, with the theory at the "zero plane" $z + t = 0$, lies in the explicit covariance of the WF and in the maximum separation of the kinematic transformations from the dynamic transformations: transformations of the reference frame are kinematic (i.e., they contain no interactions), which, for example, simplifies considerably the formation of states with a spin,[3] while the dynamic equations for the state vector Φ are obtained by examining the motion of a light-front surface with respect to a given coordinate system. One such equation is the Schrödinger equation, which follows from the analysis of the translations of the light front and which contains interactions of the form

$$i\frac{\partial\phi(\sigma)}{\partial\sigma} = \hat{H}(\sigma)\phi(\sigma), \tag{1}$$

where $\hat{H}(\sigma) = \int \hat{H}^{\rm int}(x)\delta(\omega x - \sigma)d^4x$, $\hat{H}^{\rm int}(x)$ is the interaction Hamiltonian.

The purpose of the present paper is to obtain another equation (the "angular condition") for the state vector, which follows from the analysis of the four-dimensional rotations of the light-front surface, and to determine its role in the problem for bound states. It turns out that this equation eliminates the nonphysical degeneracy of the relativistic states which appears in systems with a total angular momentum different from zero (in particular, in the Weinberg equation[6] for nonzero angular momentum).

The desired equation follows from the Tomonagi-Schwinger equation

$$i\frac{\delta\phi}{\delta\sigma(x)} = \hat{H}^{int}(x)\phi\ .$$

We examine those variations of the light-front surface $\omega x = \sigma$ which transform it to the rotated surface $\omega' x = \sigma$, where $\omega' = \omega + \delta\omega$, $\delta\omega_\mu = \epsilon_{\nu\mu}\omega_\nu$, and $\epsilon_{\nu\mu}$ are the infinitesimal rotation parameters. The total increment of the state vector $\delta\phi$ is the sum of the increments at each point of the surface: $i\delta\phi = \int \hat{H}^{\rm int}(x)\delta V(x)\Phi$. The increment of the four-dimensional volume above the point x of the surface $\omega x = \sigma$ is $\delta V(x) = \epsilon_{\mu\nu}x_\mu\omega_\nu\delta(\omega x - \sigma)d^4x$. On the other hand, $\phi(\omega) \to \phi(\omega + \delta\omega) = \phi(\omega) + \delta\phi(\omega)$, where $\delta\phi(\omega) = \epsilon_{\mu\nu}\omega_\mu \partial/\partial\omega_\mu \phi(\omega)$. Hence, we find the equation which defines the dependence of the state vector on the 4-momentum vector ω:

$$\hat{L}_{\mu\nu}(\omega)\phi = \hat{J}^{int}_{\mu\nu}\,\phi, \tag{2}$$

where

$$\hat{L}_{\mu\nu}(\omega) = i\left(\omega_\mu \frac{\partial}{\partial\omega_\nu} - \omega_\nu \frac{\partial}{\partial\omega_\mu}\right), \tag{3}$$

$$\hat{J}^{int}_{\mu\nu} = \int \hat{H}^{int}(x)(x_\mu\omega_\nu - x_\nu\omega_\mu)\delta(\omega x - \sigma)d^4x. \tag{4}$$

In addition to Eqs. (1) and (2), the state vector of a bound system satisfies the equations for the eigenvalues:

$$\hat{P}_\mu \phi^J_s = p_\mu \phi^J_s\ , \tag{5}$$

$$\hat{W}^2_\mu \phi^J_s = -\,M^2 J(J+1)\,\phi^J_s\ , \tag{6}$$

$$\hat{W}_3\,\phi^J_s = Ms\phi^J_s\ , \tag{7}$$

where $\hat{P}_\mu = \hat{P}^0_\mu + \omega_\mu \hat{H}(\sigma)$, $\hat{P}^0_\mu$ is a free operator, $p^2_\mu = M^2$, and $\hat{W}_\mu$ is the Pauli-Lyubanskiĭ vector:

$$\hat{W}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\gamma}\hat{P}_\nu \hat{M}_{\rho\gamma}\ , \tag{8}$$

where

$$\hat{M}_{\rho\gamma} = \hat{J}^0_{\rho\gamma} + \hat{L}_{\rho\gamma}(\omega)\ .$$

$\hat{J}^0_{\rho\gamma}$ is a free operator, and $\hat{L}_{\rho\gamma}(\omega)$ is given by Eq. (3). Instead of $\hat{M}_{\rho\gamma}$, the Pauli-Lyubanskiĭ vector contains the operator

$$\hat{J}_{\rho\gamma} = \hat{J}^0_{\rho\gamma} + \hat{J}^{int}_{\rho\gamma}\ ,$$

where $\hat{J}^{\rm int}_{\rho\gamma}$ is given by Eq. (4). Bearing in mind Eq. (2), however, we have replaced $\hat{J}^{\rm int}_{\rho\gamma}$ by $\hat{L}_{\rho\gamma}(\omega)$ and $\hat{J}_{\rho\gamma}$ by $\hat{M}_{\rho\gamma}$ in Eq. (8).

Equation (5) determines the momentum and spectrum of the masses, Eqs. (6) and (7) determine the angular momentum J and its projection s (in the system where $\mathbf{p} = 0$); the construction of states with a certain momentum is a purely kinematic problem, since the operator $\hat{M}_{\rho\gamma}$ which appears in Eq. (8) contains no interaction. This problem has been solved elsewhere.[3] Equation (1) determines the trivial dependence of ϕ on the "skew" time σ. What, however, does Eq. (2) determine [apart from the replacement of $\hat{J}_{\rho\gamma}$ by $\hat{M}_{\rho\gamma}$ in Eq. (8)]?

To answer this question, we note that the operator

$$\hat{A} = \omega_\mu \hat{W}_\mu \tag{9}$$

commutes with $\hat{P}_\mu$ and with $\hat{W}_\mu$; i.e., in addition to mass and spin, the states are characterized by another quantum number—the eigenvalue of the operator $\hat{A}$:

$$\hat{A}\,\phi_a = a\phi_a\,. \tag{10}$$

For example, at $J=1$ we have three states: $a=0, \pm 1$. These states are degenerate. In fact, the operator

$$\Delta\hat{J}_{\mu\nu} = \hat{M}_{\mu\nu} - \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu}(\omega) - \hat{J}^{int}_{\mu\nu} \tag{11}$$

commutes with $\hat{P}_\mu$ (the commutators of $\hat{M}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$ with $\hat{P}_\mu$ are equal), but does not commute with $\hat{A}$:

$$[\Delta J_{\mu\nu}, \hat{A}\,] = i\omega_\alpha \hat{P}_\beta(\epsilon_{\alpha\beta\mu\lambda}\,\Delta\hat{J}_{\nu\lambda} - \epsilon_{\alpha\beta\nu\lambda}\Delta J_{\mu\lambda}) + i(\hat{W}_\mu\omega_\nu - \hat{W}_\nu\omega_\mu)\,. \tag{12}$$

Therefore, the state $\Phi' = \Delta\hat{J}_{\mu\nu}\Phi_a$ is a superposition of states with different a, but with the same mass as Φ_a.

If the state vector corresponds to a momentum different from zero, then the action of the commutator (12) on the state vector gives a result different from zero, even if Eq. (2) (i.e., $\Delta\hat{J}_{\mu\nu}\Phi = 0$) holds. Therefore, Eqs. (2) and (10) are not consistent equations. The solution of Eq. (2) is a superposition of the degenerate ϕ_a states:

$$\phi = \sum_a c_a\,\phi_a\,. \tag{13}$$

Therefore, Eq. (2) eliminates the nonphysical degeneracy of the relativistic states, and since it is rewritten in the form $\Sigma c_a(\Delta\hat{J}_{\mu\nu})_{a'a} = 0$ [$(\Delta\hat{J}_{\mu\nu})_{a'a}$ are the matrix elements of the operator (11) in the ϕ_a basis], it determines (within the normalization accuracy) the coefficients in Eq. (13).

We elucidate the foregoing by the example of a two-particle WF. This WF satisfies the approximate equation obtained from Eq. (5):

$$(4(\mathbf{q}^2 + m^2) - M^2)\,\psi(\mathbf{q}, \mathbf{n}) = -\frac{m^2}{2\pi^3}\int \psi(\mathbf{q}', \mathbf{n})V(\mathbf{q}', \mathbf{q}, \mathbf{n}, M^2)\,\frac{d^3q'}{\epsilon(\mathbf{q}')}\,, \tag{14}$$

where $\mathbf{q}$ is the momentum of the particle 1 in the rest frame of the pair 1 and 2, and $\mathbf{n}$ is the direction of ω in this system.[1] Equation (14), which is written in the variables $\mathbf{k}_\perp^2 = \mathbf{q}^2 - (\mathbf{n}\cdot\mathbf{q})^2$ and $x = [1 - \mathbf{n}\cdot\mathbf{q}/\epsilon(\mathbf{q})]/2$, is identical to the Weinberg equation[6] for the WF in a system with an infinite momentum (see Ref. 4). The momentum operator has the form[3]

$$\hat{\mathbf{J}} = \frac{1}{i}\left[\mathbf{q} \times \frac{\partial}{\partial \mathbf{q}}\right] + \frac{1}{i}\left[\mathbf{n} \times \frac{\partial}{\partial \mathbf{n}}\right]. \tag{15}$$

In the limit $q \ll m$ the WF ceases to depend on $\mathbf{n}$, the term $1/i\,[\mathbf{n} \times \partial/\partial \mathbf{n}]$ can be dropped in Eq. (15), and the operator (15) becomes nonrelativistic. The operator $\hat{A}' = \mathbf{n} \cdot \hat{\mathbf{J}}$ plays the role of the operator (9). It commutes with $\hat{\mathbf{J}}$ and with the kernel V (since the kernel V is a scalar). At $J \neq 0$ Eq. (14) therefore has several solutions, which differ in the eigenvalue of the operator $\hat{A}'$, irrespective of the form of the kernel. Thus, the ground state of a deuteron is degenerate. The condition for the Fock components, which eliminates this problem, follows from Eq. (2). The explicit form of this condition depends on the original Hamiltonian $\hat{H}^{\text{int}}(x)$ and on the subsequent approximations. The condition for the two-particle WF, which is obtained within the framework of the model of two scalar particles that interact in the ladder approximation by means of a scalar massless particle exchange, will be given in a more detailed study. We note that the WF for a light front obtained previously[5] in this model and the relativistic WF of a deuteron[5] satisfy this condition.

The author wishes to thank L. A. Kondratyuk, V. E. Markushin, Dzh. M. Namyslovskiĭ, and I. S. Shapiro for fruitful and stimulating discussions.

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Translated by Eugene R. Heath
Edited by S. J. Amoretty