Calculation of the fermion determinant in chiral and supersymmetrical theories

A. I. Vaĭnshteĭn and V. I. Zakharov
Institute of Theoretical and Experimental Physics, Academy of Sciences of the USSR

(Submitted 20 January 1982)

Pis'ma Zh. Eksp. Teor. Fiz. **35**, No. 6, 258–260 (20 March 1982)

A method is proposed for integrating over fermion fields in chiral and supersymmetric theories with a real representation for fermions. The prescription allows avoiding the usual difficulties of continuation into a Euclidean space. The effect of fermion interaction, resulting from small instantons, is calculated in the supersymmetrical gluon dynamics and it is shown that the effective Lagrangian is not invariant under supersymmetry transformations.

PACS numbers: 11.30.Pb, 11.30.Rd, 12.35.Ht

Chiral theories, i.e., theories involving asymmetry between right- and left-handed particles, are widely used in the modern theory of elementary particles. It is sufficient to recall the theory of the weak interaction. Nevertheless, in chiral theories, there is a serious difficulty. In particular, continuation into a Euclidian space is impossible (the necessity for such continuation is evident in attempts to calculate nonperturbative effects such as those of instantons). Indeed, in a Euclidian space, in contrast to a Minkowski space, both the Weyl spinor ϕ as well as its Hermitian conjugate field $\phi^\dagger$ transform according to the same representation, and even the kinetic energy cannot be constructed without introducing additional fields. This problem was noted by many workers and is now explained in textbooks.[1] In the Minkowski space, the difficulties are not as evident, but they appear here as well. The introduction of both left-handed and right-handed fields is required by mass regulators and their mass term destroys gauge invariance.

In this paper, we propose a solution of these problems for chiral theories, in which fermions form a so-called real representation. More precisely, the matrices of the generators T_a in the representation and its complex conjugate must be related by a unitary transformation:

$$T^a = U(-T^a)^T U^+. \tag{1}$$

This condition includes the simplest chiral and supersymmetric theories: the gauge group SU(2) and the doublet of Weyl or triplet Majorana spinors, interacting with vector fields. We will limit ourselves to these examples.

We shall explain the prescription proposed for integrating over fermion fields for an example of the theory with the following Lagrangian:

$$L = -\frac{1}{4}(G^a_{\mu\nu})^2 + \phi^+\sigma^R_\mu iD_\mu\phi\,, \tag{2}$$

where $G_{\mu\nu}(a=1,2,3)$ is the gluon field intensity tensor, ϕ^k_a $(k, a=1,2)$ is a doublet of Weyl spinors (k is the color index), the covariant derivative D_μ is equal to $\partial_\mu + igT^aA^a_\mu$, $T^a = \tau^a/2$ for the doublet and, finally, $\sigma^R_\mu = (1, \vec{\sigma})$.

The generating functional is written as

$$Z(\eta, \eta^+) = \int DA e^{-\frac{i}{4}\int (G^a_{\mu\nu})^2 d^4x}\,\Phi(A, \eta, \eta^+) \tag{3}$$

$$\Phi(A, \eta, \eta^+) = \int D\phi D\phi^+ \exp i\int d^4x\,\{\phi^+\sigma^R_\mu iD_\mu\phi + \eta^+\phi + \phi^+\eta\}\,,$$

where η is the external source, and ϕ and $\phi^\dagger$ are Grassman variables.

Let us perform a linear substitution of variables

$$\phi^k_a = (\tau_2)^{km}(\sigma_2)_{a\beta}\,\chi^{*m}_\beta \tag{4}$$

and a corresponding substitution for $\phi^\dagger$. Then, Φ transforms into

$$\Phi(A, \eta, \eta^+) = \int D\chi D\chi^+ \exp i\int d^4x\,\{-\chi^+\sigma^L_\mu iD_\mu\chi - \chi^+(\sigma_2\tau_2\eta^*) - (\eta^T\sigma_2\tau_2)\chi\}\,, \tag{5}$$

where $\sigma^L_\mu = (-1, \vec{\sigma})$. Neither relation (3) nor (5) admits a direct continuation into Euclidian space or explicitly gauge invariant regularization by massive fields. The trick is to define the fermion determinant as the square root of the product of (3) and (5).[1)] Then, it is possible to transform to four-component notation and, therefore, integration over the fermion fields is a standard operation.[3] Explicitly,

$$\Phi^2(A, \eta, \eta^+) = \int D\Psi D\bar{\Psi} \exp i\int d^4x\,\{\bar{\Psi}\gamma_\mu iD_\mu\Psi + \bar{j}\Psi + \bar{\Psi}j\}\,, \tag{6}$$

where j and Ψ are expressed in terms of initially introduced quantities:

$$\Psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix} \qquad j = \begin{pmatrix}-\tau_2\sigma_2\eta^*\\ \eta\end{pmatrix}$$

If, for example, the effect of instantons must be found, then this effect is first calculated in the four-component notation (6) with all the necessary regularization and, then, extracting the square root of the result, we obtain an expression for the initial, two-component notation. (The uncertainty that arises in the sign, as is easily seen, can be eliminated by normalizing to perturbation theory.)

We shall present the results for another theory, namely, supersymmetrical gluon dynamics, since it is more interesting from the practical point of view. The Lagrangian is usually written in the form

$$L = -\frac{1}{4}(G^a_{\mu\upsilon})^2 + \frac{1}{2}\lambda^{aT}\gamma_0\gamma_\mu iD_\mu\lambda^a, \qquad (7)$$

where λ^a is a triplet of Majorana particles and the generators T^a are taken, correspondingly, in the triplet representation, $(T^a)_{bc} = i\epsilon_{bac}$. The Majorana spinors, as is well known, are rewritten in terms of the Weyl terms, after which the technique described above can be applied.

In order to carry out an explicit calculation, it is necessary to know the zero-order fermion modes in an instanton field. There are four of them in this case:

$$\phi^a_{1,2} = \frac{\sqrt{2}}{\pi}\frac{\rho^2}{(x^2+\rho^2)^2}\sigma^a v_{1,2}\,; \qquad v_1 = w_3 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\phi^a_{3,4} = \frac{1}{\pi}\frac{\rho}{(x^2+\rho^2)^2}\sigma^a\sigma^+_\mu x_\mu w_{3,4}\,; \qquad v_2 = w_4 = \begin{pmatrix}0\\1\end{pmatrix}, \qquad (8)$$

where $\sigma^\dagger_\mu \simeq (-i, \vec{\sigma})$, and ρ is the size of the instanton. Knowing the zero-order modes, it is possible to calculate the determinant (6), as in the usual case.[4] Extracting the root also does not present any difficulties.

We shall immediately present the results for the effective four-fermion interaction due to instantons (anti-instantons) with small dimensions:

$$L_{eff} = \frac{4\pi^4}{3}\left(\frac{2\pi}{a_s}\right)^4 \exp\left(-\frac{2\pi}{a_s}\right)\frac{d\rho}{\rho^5}$$

$$\times\left\{\lambda^{aT}\gamma_0\lambda^a\partial_\mu\lambda^{bT}\gamma_0\partial_\mu\lambda^b + \lambda^{aT}\gamma_0\gamma_5\lambda^a\partial_\mu\lambda^{bT}\gamma_0\gamma_5\partial_\mu\lambda^b - \frac{1}{2}\lambda^{aT}\gamma_0\sigma_{\alpha\beta}\lambda^b\partial_\mu\lambda^{bT}\gamma_0\sigma_{\alpha\beta}\partial_\mu\lambda^a\right\}, \qquad (9)$$

where

$$\sigma_{\alpha\beta} = \frac{1}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha).$$

In conclusion, we shall make several remarks concerning the results obtained. Condition (1), as we can see, means that the theory is actually a vectorlike theory. Indeed, in the notation (6), there is no axial current at all. On the other hand, condition (1) guarantees the absence of triangle anomalies.[5] However, triangle anomalies are also missing in the Weinberg-Salam model [or in its SU(5) and SO(10) generalizations]. Transformation to the notation (6) in the last case is impossible, and the theory is still not formulated in Euclidian space. This could possibly reflect serious difficulties.

As far as practical applications are concerned, we emphasize that the effective Lagrangian (9) explicitly breaks down supersymmetry. Indeed, supersymmetry transformations transform the fermion fields into bosons fields. On the other hand, we have obtained only the effective fermion interaction in the approximation examined. From the computational point of view, the reason for this asymmetry is obvious: Only fermion zero-order modes cause the determinant to vanish. There is a resemblance to the well-known case of U(1) symmetry, where the effective Lagrangian also explicitly breaks down symmetry,[4] and it is indeed explicitly (not spontaneously!) broken. We believe that the same is also true for supersymmetry. However, a detailed examination of this problem (together with V. A. Novikov and M. A. Shifman) falls outside the scope of this letter.

We thank V. A. Novikov and M. A. Shifman for very detailed discussions.

[1)] As noted by A. A. Migdal, squaring the partition function in order to rewrite it in the form of an integral over fermions was used in the Ising model (see Ref. 2).

---

1. P. Ramond, Field Theory, A Modern Primer, Benjamin–Cumming Publ. Co., 1981.
2. V. N. Popov, Kontinual'nye integraly v kvantovoĭ teorii polya i statisticheskoĭ fizike (Continual Integrals in Quantum Field Theory and Statistical Physics), Atomizdat, Moscow, 1976.
3. F. A. Berezin, Metod vtorichnogo kvantovaniya (Method of Second Quantization), Izd. Nauka, Moscow, 1965.
4. G. 't Hooft, Phys. Rev. **D 14**, 3432 (1976).
5. H. Georgi and S. I. Glashow **D 6**, 429 (1972).

Translated by M. E. Alferieff
Edited by S. J. Amoretty