Whitham deformations partially saturating a modulational instability in a nonlinear Schrödinger equation

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A Whitham-modulated solution of the nonlinear Schrödinger equation is constructed by the approach of Gurevich and Pitaevskiĭ. This solution corresponds to the regime of a partially saturated modulational instability. It describes a mechanism for the evolution of a monochromatic wave in the nonlinear Schrödinger equation which leads to the nucleation of a new phase and to the onset of an expanding oscillatory region.

From the standpoint of Whitham deformations,[1] the modulational instability (Refs. 2–5, for example) in the nonlinear Schrödinger (NS) equation

$$iu_t + U_{xx} + 2|u|^2 u = 0 \tag{1}$$

occurs because the characteristic velocities of the Whitham–nonlinear-Schrödinger (WNS) equations are complex. Perturbations linearized against the (locally) constant background thus grow exponentially in time. It can be suggested, however, that there exist solutions of the WNS equations that are not constant, which cause the complex part of certain characteristic velocities to vanish, and which thereby partially saturate the modulational instability.

Our purposes in the present study are to demonstrate the existence and noncontradictory nature of an approximate solution of NS equation (1) constructed with the help of Whitham deformations which partially saturate the modulational instability and to describe on the basis of this solution a mechanism for the evolution of a zero-phase solution (a monochromatic wave) of the NS equation,

$$u_0(x,t) = e^{i2t}, \tag{2}$$

which leads to the nucleation and evolution of a new phase by virtue of the WNS equations. An important part here is that time scale for the development of the new phase is on the same order of magnitude as the time scale for the development of the ordinary modulational instability.

The evolution of a new phase leads to the appearance of an oscillatory region in the solution of Eq. (1). Since one of the growth rates of the modulational instability vanishes in our case, there is an analogy with the known[6] mechanism for the nucleation of new phases in modulationally stable situations. We wish to stress that the circumstance which distinguishes our situation from Ref. 6 in terms of fundamental physics (see also Ref. 7) is that the mechanism for the nucleation of the new phase proposed below operates even when the asymptotic forms $u(x \to \pm\infty, t)$ of the solution $u(x,t)$ are the same.

The approximate solution of the NS equation which we are proposing consists of an external zero-phase solution (2) in the region $x \leq x^-(t)$ and $x \geq x^+(t)$ and an internal, Whitham-modulated, single-phase solution from (3)–(13) (see the discussion below) in the region $x^-(t) \leq x \leq x^+(t)$.

Single-phase solutions of (1) can of course be written in the form

$$u^\pm = \sqrt{f(\theta^\pm)} \exp(i\varphi^\pm), \quad \theta^\pm = x - U^\pm t, \tag{3}$$

$$f(\theta) = f_3 + (f_1 - f_3) dn^2\{\sqrt{f_1 - f_3}\theta; m\}, \quad m = (f_1 - f_2)/(f_1 - f_3), \tag{4}$$

$$\varphi_x^\pm = U^\pm/2 \mp A/f, \quad \varphi_t^\pm = -(U^\pm)^2/4 + \left(\sum_i f_i\right) \pm U^\pm A/f, \tag{5}$$

where $f_1 \geq f \geq f_2 \geq 0 \geq f_3$, $A = \sqrt{-f_1 f_2 f_3} \geq 0$, and dn is the elliptic Jacobi function. The different signs in (3)–(5) correspond to different roots ($\lambda_2 = \lambda_1^*$, $\lambda_4 = \lambda_3^*$) of the corresponding elliptical curve [determined by the expression $R^2(\lambda) = \Pi_1^4(\lambda - \lambda_i)$] (cf. Ref. 8). For the upper sign we have

$$\begin{aligned} \lambda_1^+ &\equiv \alpha^+ - i\gamma^+ = U^+/4 - \sqrt{-f_3}/2 - i(\sqrt{f_1} + \sqrt{f_2})/2, \\ \lambda_3^+ &\equiv \beta^+ - i\delta^+ = U^+/4 + \sqrt{-f_3}/2 - i(\sqrt{f_1} - \sqrt{f_2})/2, \end{aligned} \tag{6}$$

while for the lower sign we have

$$\begin{aligned} \lambda_1^- &\equiv \beta^- - i\delta^- = U^-/4 - \sqrt{-f_3}/2 - i(\sqrt{f_1} - \sqrt{f_2})/2, \\ \lambda_3^- &\equiv \alpha^- - i\gamma^- = U^-/4 + \sqrt{-f_3}/2 - i(\sqrt{f_1} + \sqrt{f_2})/2. \end{aligned} \tag{7}$$

The WNS equations for (1), (3)–(7) are (cf. Ref. 8)

$$d\lambda_i/dt + S_i(\lambda) d\lambda_i/dx = 0, \quad i = 1,2,3,4; \tag{8}$$

$$\begin{aligned} S_1 = U + 2\lambda_{12}/(1 - \mu\lambda_{32}/\lambda_{31}), \quad S_3 = U + 2\lambda_{34}/(1 - \mu\lambda_{14}/\lambda_{13}), \\ S_2 = S_1^*, \quad S_4 = S_3^*; \quad \mu \equiv E(m)/K(m), \end{aligned} \tag{9}$$

where $\lambda_{ij} \equiv \lambda_i - \lambda_j$, E and K are the complete elliptic integrals of the second and first kinds, and $m = \lambda_{21}\lambda_{43}/\lambda_{32}\lambda_{14}$.

In the oscillatory region, $0 \leq x \leq x^+(t)$, the solution of Eqs. (8) of interest here is described by the equations

$$\begin{aligned} &\lambda_1^+ \equiv \alpha^+ - i\gamma^+ \equiv \text{const}, \quad \text{Im}(S_3) = 0, \\ &\{4\beta^+ + 2[(\gamma^+)^2 - (\delta^+)^2]/(\beta^+ - \alpha^+)\}t - x = g(\beta^+, \delta^+), \end{aligned} \tag{10}$$

where $g(\beta, \delta)$ is an arbitrary smooth function of the arguments, determined by the initial conditions. We consider the very simple case $g(\beta, \delta) \equiv 0$ (cf. Refs. 6, 7, and 9). From initial condition (2) we find $\gamma^+ \equiv 1$ and $\alpha^+ \equiv 0$. In this case system (10) is invariant under the transformations $\delta^+ \to -\delta^+$ and $(\beta^+, x) \to -(\beta^+, x)$. Analysis shows that system (10) is compatible and has a unique solution with $\delta^+ \geq 0$, $\beta^+ \geq 0$ in the region $0 \leq x \leq x^+(t)$. Near the boundary $x^+ = x^+(t)$ the solution of system (10) is

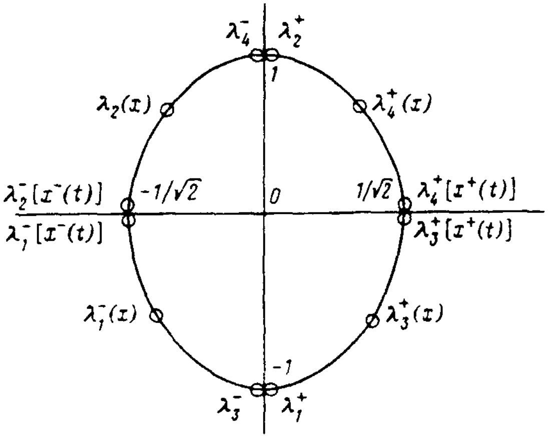

FIG. 1. Qualitative behavior of $\lambda_i^\pm$ in the complex plane.

$$x^+=4\sqrt{2}t,\quad x=x^+-x',\quad 0<x'\ll 1,$$
$$\beta^+\approx 1/\sqrt{2}-7x'/48t,\quad (\delta^+)^2\approx x'/2\sqrt{2}t, \tag{11}$$

so that at $x=x^+(t)$ solution u^+ from (3)–(5) merges continuously with u_0 from (2) on the right of $x^+(t)$. As $(x/t)\to+0$, the points $(\lambda_3^+,\lambda_4^+)$ merge with $(\lambda_1^+,\lambda_2^+)$, so that solution (3) degenerates into a soliton. Figure 1 shows the behavior of λ_i in the complex λ plane. Figure 2 is an approximate plot of the amplitude of the corresponding solution of (1) at $t\gg 1$.

In the oscillatory region, $x^-(t)\leqslant x\leqslant 0$, the solution of system (8) is given by

$$\lambda_3^-=\alpha^--i\gamma^-\equiv\text{const},\quad \text{Im}(S_1)=0,$$
$$\{4\beta^-+2[(\gamma^-)^2-(\delta^-)^2]/(\beta^--\alpha^-)\}t-x=0. \tag{12}$$

System (12) is analogous to (10). From initial condition (2) we find $\gamma^-\equiv 1$, $\alpha^-\equiv 0$. Near the boundary $x^-=x^-(t)$ the solution of system (12) is

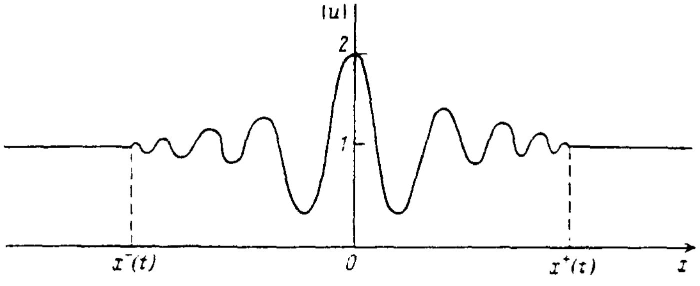

FIG. 2. Qualitative behavior of $|u|$ versus x at large t.

$$x^- = -4\sqrt{2}t, \quad x = x^- + x'', \quad 0 < x'' \ll 1,$$

$$\beta^- \approx -1/\sqrt{2} + 7x''/48t, \quad (\delta^-)^2 \approx x''/2\sqrt{2}t, \tag{13}$$

so that at $x = x^-(t)$ the solution u^- from (3)–(5) merges continuously with u_0 from (2) on the left of $x^-(t)$. As $(x/t) \to -0$, the points $(\lambda_1^-, \lambda_2^-)$ merge with $(\lambda_3^-, \lambda_4^-)$, and solution u^- from (3)–(5) degenerates into a soliton. Corresponding curves are shown in Figs. 1 and 2.

Equation (1) is invariant under a Galilean transformation, so the analysis above can be easily extended to the case of a zero-phase solution:

$$u_0(x,t) = \exp[i2\alpha x + i(2 - 4\alpha^2)t].$$

We would point out some corresponding changes in Eqs. (10)–(13): $x^\pm \to 4\alpha t + x^\pm$, $\alpha^\pm \to \alpha$, $\beta^\pm \to \alpha + \beta^\pm$, $\lambda_1^+ \to \alpha + \lambda_1^+$, $\lambda_3^- \to \alpha + \lambda_3^-$.

Single-phase WNS equations (8) generally describe two pairs of perturbations. There is accordingly the question of the validity of "discarding" one of the pairs of perturbations ($\lambda_1^+ \equiv$const or $\lambda_3^- \equiv$const) and studying the mechanism described above "in its pure form." The answer is that since we have $|dx^\pm(t)/dt| \simeq 1$ [see (11) and (13)], the time scale for the onset of the process under consideration here is on the same order of magnitude as the time scale for the onset of the modulational instability of the "discarded" mode [$(|\mathrm{Im}S_1^+|, |\mathrm{Im}S_3^-|) \simeq 1$ at the initial time]. We also note that the values $(|\mathrm{Im}S_1^+|, |\mathrm{Im}S_3^-|)$ are at a maximum only in the exterior region of the solution. They fall off to zero toward the center of the oscillatory region. It is of course clear that the overall picture of the evolution of solution (2), which simultaneously incorporates the dynamics of all the perturbations, will be more complex than in Fig. 2. The solution of this question and also its rigorous justification at large values of t (with allowance for the validity of the approach of Gurevich and Pitaevskiĭ[6] and the Whitham method[1]) requires a more complex technique than was used above (cf. the simpler situation in modulationally stable cases[7]). That problem lies outside the scope of the present paper.

From the formal standpoint, the solution constructed above should be thought of as a hypothesis regarding the leading term of an asymptotic solution of NS equation (1) with the initial conditions first considered by Karpman[10]: $u(x,0) = 1 + \epsilon(x)$, where $\epsilon(x \to \pm\infty) \to 0$ (i.e., the problem of the evolution of a local perturbation of a homogeneous background). It should be noted that the numerical results[11] (see also Ref. 12) agree qualitatively with the solution discussed by us. From this standpoint our study is a next step toward a theoretical description of the problem of Ref. 10.

During the refereeing stage of this paper, we learned of Ref. 13, on the same topic. The solution constructed in Ref. 13 is discontinuous [e.g., at the right front, $x^+(t)$] and thus distinct from (3)–(13).

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