Vortex helicoid in a superconducting cylinder in a longitudinal magnetic field

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An exact solution is derived for the problem of the penetration of a helicoidal magnetic vortex into a current-carrying superconducting cylinder in a magnetic field. The London approximation is used. The magnetic field is assumed parallel to the axis of the cylinder. A current–field diagram of the resistive state is constructed.

The resistive state of type-II superconductors is usually associated with the motion of Abrikosov vortices in these materials. In an external magnetic field, the viscous motion of the vortex lattice results in a dissipation of energy.[1] If there is no magnetic field, a dissipation arises from the penetration of closed vortices into the sample and their subsequent collapse. These vortices are induced by the self-magnetic field of the transport current.[2–4] In this picture, pinning is usually regarded as the factor governing the critical current.[1]

Recent experiments on microbridges of high-T_c superconductors,[5] however, have revealed some exceedingly high critical currents, up to 10^9 A/cm^2. These high currents have forced certain authors[4,5] to come up with alternative possibilities for the formation of the critical current. The problem of an edge barrier for the entry of a vortex ring into a current-carrying cylinder of arbitrary radius R was studied in Refs. 3 and 4. The value of the critical current, which determines the disappearance of the edge barrier, turned out to be independent of the radius, equal to the London value $j_L = cH_c/4\pi\lambda$, where H_c is the thermodynamic critical field, and λ is the London penetration depth for the magnetic field. It was shown in Ref. 4 that in bulk samples with $R \gg \lambda$, however, the width of the edge barrier is a strong function of the current. As a result, yet another characteristic current arises: $j_{c1} \simeq j_L/\varkappa$, where $\varkappa$ is the Ginzburg–Landau parameter. The current j_{c1} generates a field on the order of H_{c1} at the surface. This current is the critical current for a defect mechanism for the penetration of vortices over inhomogeneities of length scale λ (Ref. 4). At $R \ll \lambda$, a high critical current may thus be determined by a pinning which controls the motion of vortices after they have penetrated into the sample. In the case of thin superconductors with $R < \lambda$, on the other hand, the latter scale value drops out of the spatial variation of all properties,[4] and we are left with a single scale value of the current, j_L. For the typical parameter values of 1–2–3 single crystals, this current is on the order of 10^9 A/cm^2. Since the defect mechanism is ineffective for the penetration of vortices under the condition $R < \lambda$, we can take the edge barrier to be a reasonable explanation for the high critical current observed at microbridges.[5] We would expect to find similar distinctive features in the current-carrying properties of thin superconductors in a magnetic field.

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In this letter we examine the surmounting of the edge barrier by a vortex in the case in which there is an external magnetic field directed along the axis of the cylinder. We note that such a field does not affect the conditions for the penetration of a vortex ring of the self-field of the current, just as the transport current does not lower the critical field for the penetration of a linear vortex parallel to the axis of the cylinder, since force-free current–field configurations prevail in these cases.[1] On the other hand, there may be other vortex configurations, closer to the picture of the field lines, for which the edge barrier is lower and which lead to a field-dependent critical current $j_{cr}(H)$ or, alternatively, to a critical field $H_{cr}(j)$. The symmetry of the problem suggests that we look at a class of helicoidal solutions which have been discussed previously in the literature at a qualitative level.[1]

In this letter we use the London approximation to derive an exact solution for the problem of a helicoidal vortex in an ideal superconducting cylinder carrying a transport current I in an external magnetic field $\mathbf{H}_0$. By "ideal" here we mean that the cylinder contains no pinning centers or surface defects. Inside the cylinder, the vortex helicoid can be described by Maxwell's and London's equations with a special right side $\mathbf{\Phi}$ in the cylindrical coordinates (ρ,φ,z):

$$\lambda^2\,\mathbf{curlcurl}\ \mathbf{H}+\mathbf{H}=\Phi_0\frac{e_l}{e_l e_\varphi}\ \delta(\rho-r)\,\delta(z-L\varphi),\quad \mathrm{div}\mathbf{H}=0, \tag{1}$$

where Φ_0 is the flux quantum, $\mathbf{e}_l$ and $\mathbf{e}_\varphi$ are unit tangent and azimuthal vectors of the helix, respectively, r is the radius of the helix, and $2\pi L$ is its pitch. In the empty space outside the superconductor, the helicoid can be described by Maxwell's equations. The right side of $\mathbf{\Phi}$ is chosen in such a way that in an unbounded superconductor it would give rise to a flux Φ_0 directed along the z axis and would satisfy the requirement $\mathrm{div}\mathbf{\Phi}=0$ for compatibility with Maxwell's equations. The boundary conditions on Eqs. (1) are that the magnetic field $\mathbf{H}$ be continuous at the boundary of the cylinder and that the field disappear at infinity. Since the magnetic field is a potential field outside the superconductor, it can be described by a single scalar potential ψ. We thus have $\mathbf{H}=\nabla\psi$, where ψ satisfies the equation

$$\Delta\psi=0 \tag{2}$$

with the boundary condition $\psi(\rho)\to 0$ as $\rho\to\infty$.

By virtue of the symmetry of the problem, the dependence on the variables z and φ can be described by the single variable ζ: $L\zeta=z-L\varphi$. This is a periodic dependence with a period of 2π. Substituting the field components in the form of Fourier series, $H_\rho(\rho,\zeta)=\Sigma_k e^{ik\zeta}a_k(\rho)$, $H_\varphi(\rho,\zeta)=\Sigma_k e^{ik\zeta}b_k(\rho)$, $H_z(\rho,\zeta)=\Sigma_k e^{ik\zeta}c_k(\rho)$, and $\psi(\rho,\zeta)=\Sigma_k e^{ik\zeta}\psi_k(\rho)$, we find from Eqs. (1) and (2) a system of equations for the Fourier components:

$$\frac{\partial^2 a_k}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial a_k}{\partial\rho}-\frac{1+k^2}{\rho^2}\,a_k-\left(\frac{1}{\lambda^2}+\frac{k^2}{L^2}\right)a_k+i\,\frac{2k}{\rho^2}\,b_k=0, \tag{3}$$

$$\frac{\partial^2 b_k}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial b_k}{\partial\rho}-\frac{1+k^2}{\rho^2}\,b_k-\left(\frac{1}{\lambda^2}+\frac{k^2}{L^2}\right)b_k-i\,\frac{2k}{\rho^2}\,a_k=-\frac{\Phi_0}{2\pi\lambda^2 L}\,\delta(\rho-r), \tag{4}$$

$$\frac{\partial^2 c_k}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial c_k}{\partial \rho}-\frac{k^2}{\rho^2}c_k-\left(\frac{1}{\lambda^2}+\frac{k^2}{L^2}\right)c_k=-\frac{\Phi_0}{2\pi\lambda^2 r}\delta(\rho-r), \tag{5}$$

$$\frac{\partial^2 \psi_k}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial \psi_k}{\partial \rho}-\frac{k^2}{\rho^2}\psi_k-\frac{k^2}{L^2}\psi_k=0. \tag{6}$$

We also find the boundary conditions on these equations:

$$a_k(R)=\frac{\partial\psi_k}{\partial\rho}(R),\quad b_k(R)=\frac{-ik}{R}\psi_k(R),\quad c_k(R)=\frac{ik}{L}\psi_k(R),\quad \psi_k(\infty)=0.$$

Each of Eqs. (5) and (6) can be separated from the system of equations and solved in terms of modified Bessel functions $I_n(x)$ and $K_n(x)$ (Ref. 6) under the boundary conditions

$$\psi_k(\rho)=d_k K_k(|k|\rho/L), \tag{7}$$

$$c_k=\frac{\Phi_0}{2\pi\lambda^2}[\vartheta(\rho-r)P_k^k(r,\rho)+\vartheta(r-\rho)P_k^k(\rho,r)]+\frac{ikd_k}{L}\frac{I_k(\varkappa_k\rho/R)}{I_k(\varkappa_k)}K_k\left(\frac{|k|R}{L}\right), \tag{8}$$

where we have introduced $\varkappa_k^2=(R/\lambda)^2+k^2(R/L)^2$ and

$$P_l^k(x,y)=[I_l(\varkappa_k)K_l(\varkappa_k y/R)-K_l(\varkappa_k)I_l(\varkappa_k y/R)]I_l(\varkappa_k x/R)/I_l(\varkappa_k).$$

To solve Eqs. (3) and (4), we introduce the functions $f_k^{\pm}=a_k\pm ib_k$, which satisfy the equation

$$\frac{\partial^2 f_k^{\pm}}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial f_k^{\pm}}{\partial \rho}-\frac{(1\mp k)^2}{\rho^2}f_k^{\pm}-\left(\frac{1}{\lambda^2}+\frac{k^2}{L^2}\right)f_k^{\pm}=\mp\frac{i\Phi_0}{2\pi\lambda^2 L}\delta(\rho-r). \tag{9}$$

Solutions of the latter equation can also be expressed in terms of modified Bessel functions; taking account of the boundary conditions, we write them as follows:

$$\begin{Bmatrix} a_k \\ b_k \end{Bmatrix}=\frac{i\Phi_0 r}{4\pi\lambda^2 L}[\vartheta(r-\rho)[P_{k-1}^k(\rho,r)\mp P_{k+1}^k(\rho,r)]+\vartheta(\rho-r)[P_{k-1}^k(r,\rho)$$

$$\mp P_{k+1}^k(r,\rho)]]-d_k\frac{|k|}{2L}\left[\frac{I_{k-1}(\varkappa_k\rho/R)}{I_{k-1}(\varkappa_k)}K_{k-1}\left(\frac{|k|R}{L}\right)\right.$$

$$\left.\pm\frac{I_{k+1}(\varkappa_k\rho/R)}{I_{k+1}(\varkappa_k)}K_{k+1}\left(\frac{|k|R}{L}\right)\right]. \tag{10}$$

To find the remaining unknown coefficient d_k, we use Maxwell's equation $\mathrm{div}\mathbf{H}=0$, which takes the following form in Fourier components:

$$\frac{\partial a_k}{\partial\rho}+\frac{a_k}{\rho}-\frac{ikb_k}{\rho}+\frac{ik}{L}c_k=0. \tag{11}$$

Hence

$$d_k=\frac{i\Phi_0 r}{2\pi\lambda^2|k|}$$

$$\times\frac{[I_{k+1}(\varkappa_k r/R)/I_{k+1}(\varkappa_k)-I_{k-1}(\varkappa_k r/R)/I_{k-1}(\varkappa_k)]}{\varkappa_k I_k(\varkappa_k)\left[\frac{K_{k-1}(|k|R/L)}{I_{k-1}(\varkappa_k)}+\frac{K_{k+1}(|k|R/L)}{I_{k+1}(\varkappa_k)}\right]+2|k|(R/L)K_k(|k|R/L)}. \tag{12}$$

Expressions (7), (8), (10), and (12) give a comprehensive description of the structure of the vortex. The free energy of the vortex per unit length of the cylinder, along the z axis, defined in the usual way[1] (a constant part which is independent of the position of the vortex is eliminated), reduces to

$$F=\frac{1}{8\pi}\int dV\mathbf{H}\cdot\mathbf{\Phi}=\frac{\Phi_0}{8\pi}\left[H_z(\rho=r,0)+\frac{r}{L}H_\varphi(\rho=r,0)\right]. \tag{13}$$

In evaluating (13) we should cut off the logarithmic divergence (standard in the London theory) at the length scale ξ, the superconducting coherence length.

To analyze the behavior of the vortex in an external magnetic field in the case with a transport current, we need to calculate the Gibbs energy of the system (per unit length):

$$G=F-\Delta W_I-\Delta W_H, \tag{14}$$

where ΔW_I is the work performed by the source of direct current in moving the vortex, and ΔW_H is the work performed by the field source. By virtue of the cylindrical geometry of the problem, the latter work can be found from

$$\Delta W_H=\frac{1}{4\pi}\int \mathbf{H}\cdot\mathbf{H}_0 dV=\frac{H_0\Phi_H(r)}{4\pi}, \tag{15}$$

where $\Phi_H(r)$ is the flux through a helical vortex of radius r along the z axis. Using (8), we find

$$\Phi(r)=\Phi_0\,[1-I_0(r/\lambda)/I_0(R/\lambda)]. \tag{16}$$

The work performed by the current source per unit length of the cylinder can be calculated in the spirit of Ref. 7 from $\Delta W_I=I\Phi_I(r)/c$, where $\Phi_I(r)$ is the flux associated with the motion of a vortex from the circuit of the dc source across the surface of the sample, on the basis that the current source generates an emf of the opposite sign, $\mathscr{E}=(d\Phi/dt)/c$, as the magnetic flux penetrates into the sample, in order to maintain a constant transport current I. Using (10) to calculate the magnetic flux

$$\Phi_I(r)=(\Phi_0/2\pi L)[1-(r/R)I_1(r/\lambda)/I_1(R/\lambda)], \tag{17}$$

we find the following final expression for the Gibbs potential of a helical vortex in a cylinder carrying a current I immersed in a parallel field H_0:

$$G=F-\frac{H_0\Phi_0}{4\pi}\left[1-\frac{I_0(r/\lambda)}{I_0(R/\lambda)}\right]-\frac{I\Phi_0}{2\pi cL}\left[1-\frac{rI_1(r/\lambda)}{RI_1(R/\lambda)}\right]. \tag{18}$$

A plot of the Gibbs potential versus the helix radius r reveals the existence of an edge barrier to the penetration of the helicoid (Fig. 1). The critical parameters (of the current or the field) for the spontaneous penetration of a vortex are determined by the condition

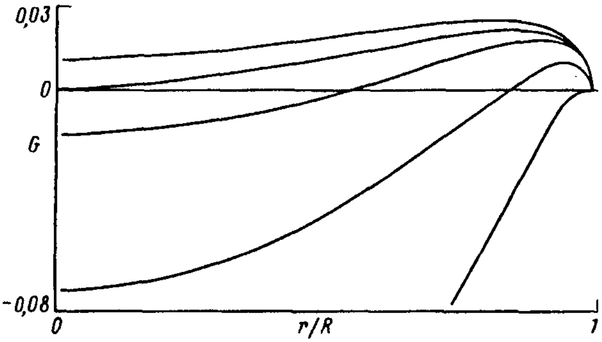

FIG. 1. The Gibbs energy of a helicoidal vortex, in units of $(\Phi_0/4\pi\lambda)^2$, versus the radius of the vortex at a transport current density $j=0.5j_L$. The radius of the cylinder is $R=0.5\lambda$; the height of a turn of the helix is $\pi\lambda$. The different curves correspond to different values of the external magnetic field.

under which $G(r)$ reaches a maximum at the surface of the cylinder, i.e., $\partial G/\partial r|_{R-\xi}=0$. In the latter equation, we have formally cut off the logarithmic divergence typical of the London theory at the length scale ξ.

Near the surface of the cylinder, under the condition $R-r\ll R$, λ , the free energy in (13) is dominated by

$$F\simeq(\Phi_0/4\pi\lambda)^2\sqrt{1+s^2}\,\ln[2(R-r)/\xi], \tag{19}$$

where $s=R/L$ is the slope of the helicoid with respect to the z axis. Substituting (19) into (18), and using the condition written above, we find the critical field $H_{\rm cr}(j,s)$, which depends on the current and on the pitch of the helix:

$$H_{\rm cr}(j,s)=H_c[I_0(R/\lambda)/I_1(R/\lambda)](\sqrt{1+s^2}-sj/j_L). \tag{20}$$

Here j is the value of the transport current density at the surface, and $H_c=\Phi_0/2\pi\lambda\xi\sqrt{2}$ is the thermodynamic critical field. This expression is described by a curve with a minimum with respect to s. We can thus find an optimum helicoid, for which the critical current (or field) for penetration is at a minimum and which is thus the first to enter the sample. Minimizing (20), we find the optimum helix pitch to be

$$L=R\sqrt{(j_L/j)^2-1}. \tag{21}$$

The corresponding values of the critical field $H_{\rm cr}$ and the critical current $j_{\rm cr}$ for the penetration of the optimum helicoid satisfy the simple relation

$$[I_1(R/\lambda)/I_0(R/\lambda)]^2(H_{\rm cr}/H_c)^2+(j_{\rm cr}/j_L)^2=1. \tag{22}$$

While the direction of the optimum vortex in the case $R\gg\lambda$ is the same as the field lines of the total magnetic field at the surface, in the case $R<\lambda$ the inclination of the field lines is steeper by a factor of $(2\lambda/R)^2$ than that of the vortex.

Relation (22) limits the region of a dissipationless state of an ideal current-carrying superconductor in a longitudinal magnetic field in a current–field diagram. We see that in the case of a thin superconductor, with $R \ll \lambda$, the field dependence of the critical current is weaker by a factor of $(2\lambda/R)$ than that in the case of a thick superconductor.

When we consider surface defects with dimensions on the order of λ (such defects are possible in the case $R \gg \lambda$), we naturally find a lowering of the critical parameters for the entrance of vortices into thick samples. In this case we may find distinctive plots of $j_{cr}(H)$ in accordance with a dependence of the width of the edge barrier on j and H . This possibility is a topic for further study. In the case of thin samples, on the other hand, as in the penetration of vortices of the self-field of the current,[4] the only length scale for the barrier width is the size of the sample. We would thus expect that surface defects of size $\delta \ll R$ would not have any strong effect on the diagram of the resistive state, (22).

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