Nonequilibrium noise in a mesoscopic conductor: A microscopic analysis

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Current fluctuations are studied in a mesoscopic conductor using the nonequilibrium Keldysh technique. A general expression is derived for the fluctuations in the presence of a time-dependent voltage, which is valid for arbitrary relation between voltage and temperature. Two limits are then considered: a voltage pulse and a dc voltage. A voltage pulse causes phase-sensitive current fluctuations, for which we derive microscopically an expression periodic in $\int V(t)dt$ with a period h/e. Applied to current fluctuations in Josephson circuits caused by phase slips, it gives an anomalous contribution to the noise with a logarithmic singularity near the critical current. In the dc case we obtain quantum-to-classical shot noise reduction factor 1/3, in agreement with recent results of Beenakker and Büttiker.

Introduction. Recently, the Landauer approach to electric transport[1] was extended to describe current fluctuations.[2–4] The central result formulated for a single channel conductor with a transmission coefficient T is that at zero temperature the current noise is given by $(e^2/h)T(1-T)eV$, where V is the voltage drop across the system. Thus the quantum noise comes to be a factor $1-T$ below the classical shot noise level, eI. Beenakker and Büttiker[5] generalized this picture to a mesoscopic conductor, where there are many conducting channels with a distribution of transmission constants T_n. The noise is a sum of contributions of separate channels: $\Sigma_n T_n(1-T_n)e^3V/h$. Since in the limit of large, dimensionless conductance G the distribution of T_n is provided by the random matrix theory in the universal form, $P(T)dT=Gdx$, where $T=1/\cosh^2 x$, we obtain for the noise $\frac{1}{3}Ge^3V/h$; i.e., the theory predicts universal quantum-to-classical noise ratio 1/3. Another approach was recently developed by Nagaev,[6] who used the kinetic equation. With this technique he also obtained the factor 1/3.

In order to better understand the relationship of these results with the conventional many-body methods, it would be of interest to perform a microscopic calculation. In this paper we will study noise using the nonequilibrium Keldysh technique and we will derive a more general formula for current fluctuations caused by a time-dependent voltage. In the dc limit our results agree with those found by other methods, and they give a generalization to the ac case. We will study the current–current correlation $\langle\langle I(t_1)I(t_2)\rangle\rangle$ in a mesoscopic conductor under a slowly varying voltage and we will show that it is a periodic function of $\Phi_{12}=c\int_{t_1}^{t_2}V(t)dt$ with a period $\Phi_0=hc/e$, the single-electron flux quantum. We will analyze current fluctuations in a metal caused by a short voltage pulse or, equivalently, by a varying magnetic flux. In this case the current fluctuations show a phase sensitivity (nonstationary Aharonov–Bohm effect) which is lacking in the dc case.[7] Recently, for a single-channel conductor geometry the phase-sensitive noise was expressed in terms of the transmission coefficient[8] T. We derive it below for a mesoscopic conductor within the Keldysh formalism and show that it is given by the normal metal conductance G reduced by the same factor 1/3.

We do the calculation for a cylindrical contact between two ideal leads. As a simple mesoscopic point of view, we assume diffusive regime inside the contact and completely ignore phase breaking effects, except for the temperature. Also, we treat electrons as noninteracting fermions. This approach somewhat restricts the range of applicability of our results, since interactions in real systems are important. However, even for interacting fermions, our calculation remains valid as long as the Fermi liquid picture holds.

Main results. For the calculation we use two arbitrary sections x_1, x_2 of the contact, and find the current–current correlator $S_{x_1,x_2}(t_1,t_2)=\langle\langle\hat{I}_{x_1}(t_1)\hat{I}_{x_2}(t_2)\rangle\rangle$. It is clear in the calculation that $S_{x_1,x_2}(t_1,t_2)$ does not depend on the choice of the sections x_1, x_2, when the time scale τ^* on which the voltage $V(t)$ varies is much longer than the time of diffusion through the contact. Under the assumption that τ^* is large, for a cylindrical contact our result is

$$S(t_1,t_2)=S_{\rm eq}(t_1-t_2)\left(1-\frac{1}{6}\,|1-e^{i\phi_{12}}|^2\right),\quad \phi_{12}=\frac{2\pi c}{\Phi_0}\int_{t_1}^{t_2}V(t)dt, \tag{1}$$

where $S_{\rm eq}(t_1-t_2)$ is the correlation in equilibrium

$$S_{\rm eq}(t)=\int e^2G\omega\,\coth\frac{\omega}{2T}\,e^{-i\omega t}\,\frac{d\omega}{4\pi^2}=-\mathrm{Re}\,\frac{e^2GT^2}{2\,\sinh^2\pi T(t+i0)}\,. \tag{2}$$

The correlator exhibits Fermi anticorrelation in the time domain, since it is explicitly negative at any $t_1\neq t_2$. We emphasize, however, that the corresponding fluctuation of the transmitted charge is positive because of the compensation with a singularity in $S_{\rm eq}(t_1,t_2)$ at $t_1=t_2$. In addition, since the second term in Eq. (1) gives a positive contribution to the correlator, the inequality $S(t_1,t_2)\geqslant S_{\rm eq}(t_1-t_2)$ holds, which means that the excess noise is strictly positive.

At a constant applied voltage, $\phi_{12}=(e/h)V(t_2-t_1)$, the correlator $S(t_1,t_2)$ is a function only of t_1-t_2, and thus it can be characterized by a spectral density

$$S_\omega=\frac{1}{6}\,(4S_\omega^{\rm eq}+S_{\omega+eV}^{\rm eq}+S_{\omega-eV}^{\rm eq}), \tag{3}$$

where $S_\omega^{\rm eq}=e^2G\omega\,\coth(\omega/2T)$ is the equilibrium Nyquist noise spectrum. At low frequency $\omega\ll T$ (eV) the noise is then given by

$$S_0=\frac{1}{3}\,e^2G\left(4T+\coth\frac{eV}{2T}\right).$$

In the limit $T=0$ this gives $S_0=\frac{1}{3}e^2GV$, the well-known quantum shot noise result.[5,6]

In addition to the dc case, Eq. (1) allows us to study noise in any ac setup, e.g., current fluctuations due to varying magnetic flux or due to a voltage pulse. Physically, in this case the system can be realized as a normal metal ring in an ac magnetic field, or as a shunt resistor of a superconducting circuit in the regime of the nonstationary Josephson effect. We consider a step-like time dependence of the flux, which corresponds to a voltage pulse, and for $T=0$ we derive an expression for the fluctuations of the transmitted charge:

$$\langle\langle\delta Q^2\rangle\rangle=\frac{1}{3}\,e^2G\left(\frac{\Phi}{\Phi_0}+\frac{2}{\pi^2}\sin^2\pi\,\frac{\Phi}{\Phi_0}\,\ln\frac{t_0}{\tau^*}\right), \tag{4}$$

where $\delta Q=\int_{-t_0}^{t_0}\hat{I}(t')dt'$ is the charge transmitted over the interval $-t_0<t<t_0$, Φ is the height of the flux step, and τ^* is the duration of the step, which is assumed to be much shorter than t_0. A similar expression was derived recently for a single channel conductor.[8] In Eq. (4) the first term corresponds to the $\omega=T=0$ noise [Eq. (3)] integrated over time, since the flux and the voltage are related, $V(t)=-(1/c)\Phi(t)$. The second Φ_0–periodic term with an infrared logarithmic divergence corresponds to the nonstationary AB effect.[7] We intend to observe it in a Josephson circuit with a shunt, where it causes an anomalous contribution to the noise in the shunt, diverging as I approaches I_c.

General formalism. Let us turn to the calculation. In the current–current correlator $\langle\langle\mathbf{j}(\mathbf{r}_1,t_1)\mathbf{j}(\mathbf{r}_2,t_2)\rangle\rangle$ we take the times t_1 and t_2 in the different branches of the Keldysh contour, and write $\langle\langle j_1j_2\rangle\rangle$ in terms of the functions G^{+-} and G^{-+} as $\langle\langle j_1j_2\rangle\rangle=\mathrm{Tr}(j_1G_{12}^{+-}j_2G_{21}^{-+})$. The trace Tr means integration over inner momenta and energy, summation over spin indices, and averaging over configurations of the random potential. Expressing the Fourier transform of the correlation function in terms of the retarded and advanced Green's functions G^R and G^A and the Keldysh function F, we obtain

$$\langle\langle jj\rangle\rangle_{k,\omega}=\frac{1}{4}\,\mathrm{Tr}\,(j_{k,\omega}Fj_{-k,-\omega}F)+\frac{1}{2}\,\mathrm{Tr}\,(j_{k,\omega}G^Rj_{-k,-\omega}G^A). \tag{5}$$

The functions $G^{R(A)}$ are familiar: $G_{\epsilon,p}^{R(A)}=(\epsilon-\xi_p\pm i/2\tau)^{-1}$. The function F satisfies Dyson's equation.[9] If the mean free path $l\ll L$ is short, it reduces to the diffusion equation for the quantity $(i/\tau)\tilde{s}(r,t_1,t_2)$, which is defined according to $F=G^R\tilde{s}-\tilde{s}G^A=(i/\tau)G^R\tilde{s}G^A$. Following the usual procedure,[10,11] we treat the vertex $\tilde{s}$ as a two-time diffusion[11,12] which satisfies the equation

$$(\partial_{t_+}-D\tilde{\nabla}^2)\tilde{s}(\mathbf{r},t_1,t_2)=0,\quad t_+=\frac{1}{2}\,(t_1+t_2), \tag{6}$$

together with the condition at the boundary with the leads:

$$\tilde{s}(\mathbf{r},t_1,t_2)=\frac{i}{\tau}\,\tilde{s}_0(t_1-t_2), \tag{7}$$

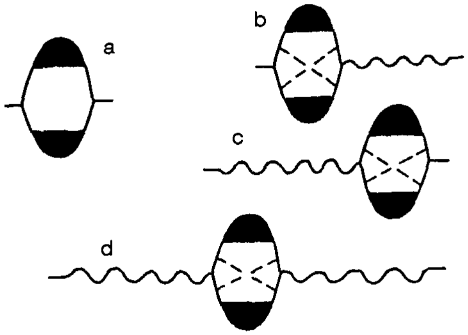

FIG. 1. Graphs of the current fluctuations. (a) The local fluctuations; (b,c,d) the nonlocal fluctuations. Black vertices represent the Keldysh function $(i/\tau)\tilde{s}$ [see Eqs. (6) and (7)]; the wavy lines are diffusions, and each box is dressed by impurity lines to form a Hikami vertex.[13]

where $\tilde{s}_0(t)=\int e^{-i\epsilon t}\tanh(\epsilon/2T)(d\epsilon/2\pi)$; i.e., the leads serve as reservoirs of equilibrium electrons. The vector potential $A(\mathbf{r},t)$ enters Eq. (6) through $\tilde{\nabla}=\nabla-i(e/c)A(\mathbf{r},t_1)+i(e/c)A(\mathbf{r},t_2)$. Initially, at $t_+=-\infty$, $\tilde{s}(\mathbf{r},t_1,t_2)=\tilde{s}_0(t_1-t_2)$ everywhere in the contact. Note that our definition of the diffusion differs from that of Ref. 12, because $\tilde{s}$ is the vertex of the function F, and thus it is a solution of the kinetic equation, which in this case reduces to Eq. (6) in a conventional way.

Before we discuss averaging over the disorder, let us evaluate simple diagram in Fig. 1a. This contribution to the current correlation is local, since it decays at distances $\gg l$. We will therefore write it in terms of the conductivity σ as

$$\langle\langle \delta j^{\alpha}(r_1,t_1)\delta j^{\beta}(r_2,t_2)\rangle\rangle = -\frac{1}{2}\, e^2\sigma \mathrm{Re}[\tilde{s}(r,t_1+i0,t_2)\tilde{s}(r,t_2-i0,t_1)]\ \delta_{\alpha\beta}\ \delta(r_1-r_2), \tag{8}$$

with the regularization at equal times which follows from the second term in Eq. (5). Now we can dress any current vertex in Fig. 1a by a diffusion ladder, which gives the other three diagrams of the figure. The analytic elements are standard: the factor $-iD\mathrm{k}e\pi\nu_0\tau$ corresponds to the current vertex and the factor $(\pi\nu_0\tau)^{-1}/(-i\omega\tau+Dk^2\tau)$ corresponds to the diffusion. In the limit $\tau^*\gg L^2/D$ we set $\omega=0$ in the diffusons, and then the sum of all contributions to the noise, the nonlocal contributions (parts b, c, and d of Fig. 1) and the local contribution (Fig. 1a), is given by

$$\langle\langle j^{\alpha}(t_1)j^{\beta}(t_2)\rangle\rangle_k = \left[\delta_{\alpha\alpha'}-\frac{k_\alpha k_{\alpha'}}{k^2}\right]\langle\langle \delta j^{\alpha'}(t_1)\delta j^{\beta'}(t_2)\rangle\rangle_k \left[\delta_{\beta\beta'}-\frac{k_\beta k_{\beta'}}{k^2}\right]. \tag{9}$$

We can verify that, because of the projector form of the expressions in the brackets [...], the correlation function of the currents in terms of the two arbitrary sections is independent of the choice of the sections.

The result (9) allows a simple and natural interpretation in terms of a diffusion equation with a random current source, $\partial n/\partial t=D\nabla^2 n-\nabla\delta j$. Using Eq. (8) for the fluctuations of δj, we obtain an expression for the fluctuation of the total current $j=-D\nabla n+\delta j$, equal to the sum of the graphs in Fig. 1, which in the low-frequency limit reduces to Eq. (9).

Cylindrical geometry. The rest of our discussion depends on the specific shape of the contact. Let us consider a cylindrical contact of length L between two ideal leads which serve as reservoirs of equilibrium electrons.

Let us solve Eq. (6) assuming that the time scale τ^* on which the field varies is longer than the diffusion time, $\tau^* \gg L^2/D$:

$$\tilde{s}(x,t_1,t_2)=\tilde{s}_0(t_2-t_1)\left[1-\frac{x}{L}+\frac{x}{L}e^{i\phi(t_2)-i\phi(t_1)}\right]e^{-i\phi(x,t_2)+i\phi(x,t_1)}, \tag{10}$$

where x is the coordinate along the cylinder axis,

$$\phi(x,t)=\frac{2\pi}{\Phi_0}\int_{-\infty}^{x}A(x',t)dx',$$

$0<x<L$ and $\phi(t)=\phi(L,t)$. The solution satisfies the boundary conditions (7) at $x=0$ and $x=L$. With $\tau^* \gg L^2/D$, we obtain Eq. (10) by ignoring the time derivative of $\tilde{s}(\mathbf{r},t_1,t_2)$ in Eq. (6) with respect to the space derivative. The vector potential in Eq. (6) can then be eliminated by a gauge transformation. We should bear in mind here that the gauge transformation changes the boundary conditions for the new function $\tilde{s}'$:

$$\tilde{s}'|_{x=L}=\tilde{s}_0(t_2-t_1)e^{i\phi(t_2)-i\phi(t_1)}.$$

The transformed function $\tilde{s}'$ satisfies the standard Laplace's equation which can easily be solved.

In the cylindrical geometry the quantity k^{-2} in Eq. (9) should be interpreted as the Green's function $\mathscr{D}(\mathbf{r}_1,\mathbf{r}_2)$ of the Laplace's operator ∇^2. According to Eq. (10), the source fluctuations (8) are uniform in every section of the cylinder, i.e., they depend only on x. Thus the problem becomes effectively one-dimensional, and we can write

$$\mathscr{D}(x_1,x_2)_{\omega=0}=\begin{cases} x_1(L-x_2)/L, & x_1<x_2 \\ x_2(L-x_1)/L, & x_2<x_1 \end{cases}. \tag{11}$$

We substitute $\partial_{x_1}\mathscr{D}(x_1,x_2)\partial_{x_2}$ for $k^x k^x/k^2$ in Eq. (9), and readily obtain (1).

The ac voltage noise. Let us now consider the fluctuations caused by a voltage pulse of duration $\tau^* \ll \hbar/T$. Setting $T=0$ in Eq. (1), we obtain

$$S(t_1,t_2)=-\frac{1}{3}e^2G\ \mathrm{Re}\ \frac{|1-e^{i\phi_{12}}|^2}{4\pi^2(t_2-t_1+i0)^2}. \tag{12}$$

Let us calculate the fluctuation of the charge transmitted during the time interval $-t_0<t<t_0$:

$$\langle\langle \delta Q^2\rangle\rangle=\int_{-t_0}^{t_0}\int_{-t_0}^{t_0}S(t_1,t_2)dt_1dt_2.$$

The logarithmically diverging term in Eq. (4) is obtained from (12) by integrating over the times t_1 and t_2 before and after the voltage pulse ($t_0/\tau^* \to \infty$):

$$\langle\langle \delta Q^2\rangle\rangle_{\log}=\frac{e^2G}{3\pi^2}\sin^2\frac{\pi\Phi}{\Phi_0}\left[\int_{-t_0}^{-\tau^*}dt_1\int_{\tau^*}^{t_0}\frac{dt_2}{(t_2-t_1)^2}+(t_1\leftrightarrow t_2)\right]$$

$$=\frac{e^2G}{3\pi^2}\,2\,\sin^2\frac{\pi\Phi}{\Phi_0}\ln\frac{t_0}{\tau^*}\,. \qquad (13)$$

The contribution proportional to Φ/Φ_0 is extracted from almost identical times t_1 and t_2. We accordingly integrate (12) over $t=\frac{1}{2}(t_1+t_2)$ and $t'=t_2-t_1$. Assuming that the time dependence of ϕ is smooth and monotonic, we write $\phi_{12}=\dot{\phi}(t)t'$, take the integral over t', and with $\dot{\phi}(t)>0$ find the first term in Eq. (4). At finite temperature, Eqs. (4), (12), and (13) hold for $t_0\leq\hbar/T$; otherwise, t_0 must be replaced by $\hbar/T$.

Let us consider a Josephson junction shunted by a normal metal resistor. If the current is set above the critical, $I>I_c$, the voltage across the resistor oscillates with the Josephson frequency $\omega=2e\bar{V}/\hbar$:

$$V(t)=R\,\frac{I^2-I_c^2}{I+I_c\cos\omega t}, \qquad (14)$$

where $R^{-1}=(e^2/h)G$, and $\bar{V}=R\sqrt{I^2-I_c^2}$. At $I-I_c\ll I_c$, the signal (14) corresponds to a periodic sequence of steps in the flux, of duration $\tau^*=RI_c$ and of height $\Phi=\frac{1}{2}\Phi_0$ each. We will use Eq. (4), which was derived for a single step, to estimate the current noise in the shunt caused by the signal (14). The step height Φ corresponds to the half-period in Eq. (4), and thus we obtain a logarithmically diverging noise associated with each step. For the low-frequency noise spectrum this gives

$$S_{\omega=0}=\frac{1}{3}\,e^2G\bar{V}\left\{1+\frac{1}{2\pi^2}\ln\frac{RI_c}{\bar{V}}\right\}, \qquad (15)$$

where the temperature is assumed to be small, $T\ll e\bar{V}$. The noise (15) is anomalously large, since it exceeds the shot noise level, $(1/3)e^2G\bar{V}$, by a logarithmic factor, which diverges as $I\rightarrow I_c$.

Experimentally, the noise (15) is more easily observed in the temperature region[14] $e\bar{V}<T<eRI_c$, where it gives a correction to the thermal noise.[15] For such temperatures, $\ln RI_c/\bar{V}$ in Eq. (15) should be replaced by $\ln eRI_c/T$.

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