Heat transfer to laminar flow across plates and cylinders of various cross sections

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Local and total Nusselt numbers were calculated for a potential flow across a flat plate for the case of large or moderate Péclet numbers (Pe). The method of solution is based on multiple applications of the Wiener–Hopf method by solving the integral equation for the power source function. Conformal transforms provide Nusselt numbers for cylinders of various cross sections. The accuracy of the method is on the order of $\exp(-2Pe)$.

The problem of heat transfer for a body moving through a liquid is a classical fluid mechanics problem (see the review in Ref. 1). Boussinesq[2] first used the conformal transformation to reduce the problem for the cylinder in a potential flow of an inviscid liquid to the problem for a thin plate in a uniform flow. This simple case, however, has no exact solution.

The model of an inviscid liquid can be applied to the flow of liquids with a very low Prandtl number (like liquid metals). Since the Prandtl number is the ratio of the momentum to the thermal diffusivity, the thermal-boundary-layer thickness is much greater than that of the momentum layer. Small Prandtl numbers correspond to large Reynolds numbers (for large or moderate Péclet numbers). The flow is laminar only for flows past slender bodies.

The second example is the flow past bodies with a free surface (like a gas bubble[3]). If a laser beam is used as a power source (e.g., by welding), it burns a long, almost cylindrical cavity (keyhole) through the metal to be welded (see, e.g., Ref. 4). The temperature of the keyhole surface is equal to the evaporation temperature of the metal T_v. In the case of electron beam welding the electrons directly heat the metal and the keyhole cross section reproduces the shape of the electron beam (circular, elliptic or other shape). The surface of the keyhole is a free surface for liquid metal and the flow remains laminar at large Reynolds numbers for a bluff (e.g., circular) cylinder.[5] In the present paper we derive analytical formulas for the heat transfer to the fluid at large and moderate Péclet numbers.

First, we consider a thin plate of length L (in the x direction) in a flow with constant velocity $v_x = v_0$, $v_y = 0$. The thermal conductivity equation (in the rest frame of the plate) is

$$v\nabla T - \kappa\Delta T = 0, \tag{1}$$

where κ is thermal diffusivity of the liquid. The boundary conditions correspond to a constant temperature, $T=T_v$, at the plate and to the ambient temperature T_0. We use dimensionless variables: $T=(T-T_v)/(T_v-T_0)$, and the length unit is $l_0=v_0/(2\kappa)$. The dimensionless length of the plate is known as the Péclet number (based on $L/2$): $a=Lv_0/(2\kappa)=Pe$.

The dimensionless, thermal conductivity equation is

$$2\frac{\partial T}{\partial x}-\Delta T=2h_p(x)\delta(y). \tag{2}$$

Here $h_p(x)$ is the heat flux distribution along the surface (the local Nusselt number) for one side of the plate. Note that $h_p(x)$ has nonzero value at the plate only. Our purpose is to find $h_p(x)$ that gives $T=1$ at the plate ($0<x<a$ and $y=0$) and $T=0$ at infinity.

The substitution

$$W(x,y)=T(x,y)e^{-x}, \quad \rho(x)=h_p(x)e^{-x} \tag{3}$$

transforms the thermal conductivity equation into the Helmholtz equation

$$W-\Delta W=2\rho(x)\delta(y) \tag{4}$$

with the boundary conditions $W=e^{-x}$ at $0<x<a$ and $y=0$. The Green's function of the Helmholtz operator is $k_0(r)/(2\pi)$ (see, e.g., Ref. 6), where $K_0(r)$ is the modified Bessel function.[7] This function provides an integral equation for $\rho(x)$

$$\int_0^a \frac{K_0(|x-s|)}{\pi}\rho(s)ds=e^{-x} \quad \text{at} \quad 0<x<a. \tag{5}$$

Unfortunately, the right side is a known function at the finite interval $(0,a)$ only and it is impossible to use directly the Fourier transformation to solve this equation. We can now use the Wiener–Hopf method (see, e.g., Ref. 6) to obtain the exact solution in the case of an infinite a and an approximate solution in the case of a large, finite a.

In the case of an infinite a the function $\rho(x)=\rho_0(x)$ is equal to zero at $x<0$. We denote as $f^+(x)=e^{-x}$ the right side of the integral equation (5) at $x>0$. The value of the right side at $x<0$ is an unknown function $g^-(x)$. The index + (or −) shows that the function is not zero at a positive (negative) x only. The convolution theorem provides an algebraic equation for the Fourier transforms of $\rho_0(k)$, $f^+(k)$, and $g^-(k)$ (all Fourier transforms appearing in the text are listed in Ref. 8):

$$\frac{\rho_0(k)}{\sqrt{1+k^2}}=f^+(k)+g^-(k). \tag{6}$$

The functions $\rho_0(k)$ and $f^+(k)=(1-ik)^{-1}$ are analytical functions of complex k in the upper half-plane and $g^-(k)$ is an analytical function in the lower half-plane. In accordance with the Wiener–Hopf method, we must collect all terms which are analytical in the upper (lower) half-plane on the left (right) side. We find

$$\frac{\rho_0(k)}{\sqrt{1-ik}}-\frac{\sqrt{2}}{1-ik}=g^-(k)\sqrt{1+ik}+\frac{\sqrt{1+ik}-\sqrt{2}}{1-ik}. \tag{7}$$

To calculate $\rho_0(x)$, we must retain all terms analytical in the upper half-plane only. We find

$$\rho_0(k)=\frac{\sqrt{2}}{\sqrt{1-ik}} \quad \text{and} \quad \rho_0(x)=e^{-x}\sqrt{\frac{2}{\pi x}}. \tag{8}$$

The next step is the solution for finite a. The function $\rho_0(x)$ satisfies Eq. (5), but it is not equal to zero at $x>a$. However, we can improve the solution by using the Wiener–Hopf method in the shifted coordinate system. In the following text the index $+$ $(-)$ denotes the function that is not equal to zero at $x>a$ $(x<a)$ only. The function $\rho_0(x)$ is the sum of two functions: $\rho_0^+(x)=\rho_0(x)\Theta(x-a)$ and $\rho_0^-(x)=\rho_0(x)-\rho_0^+(x)$. If we cut $\rho_0(x)$ at $x>a$, we must add a function $\rho_1^-(x)$ to compensate for the change on the left side in (5) at $x<a$. The change on the right side in Eq. (5) at $x>a$ is an unknown function $g^+(x)$. The corresponding equation for the Fourier transforms is

$$\frac{\rho_1^-(k)-\rho_0^+(k)}{\sqrt{1+k^2}}=g^+(k) \tag{9}$$

or

$$\frac{\rho_1^-(k)+\rho_0^-(k)}{\sqrt{1+ik}}=\frac{\sqrt{2}}{2\sqrt{1+k^2}}+g^+(k)\sqrt{1-ik}. \tag{10}$$

The left side of the last equation is an analytical function in the lower half-plane. The last term on the right side is an analytical function in the upper half-plane. Now we must decompose the first term on the right side as a sum of two functions analytical in the upper or lower half-plane. This term is the Fourier transform of $\sqrt{2}K_0(|x|)/\pi$. If we set this function to zero at $x>a$, the Fourier transform will be an analytical function in the lower half-plane. The Fourier transform of the remaining terms is an analytical function in the upper half-plane. Taking into account all terms analytical in the lower half-plane, we obtain

$$\rho(k)=\rho_1^-(k)+\rho_0^-(k)=\frac{\sqrt{2(1+ik)}}{\pi}\int_{-\infty}^{a}K_0(|x|)e^{ikx}dx. \tag{11}$$

The inverse Fourier transform gives $\rho(x)$ and the local Nusselt number (3):

$$h_p(x)=\sqrt{\frac{2}{\pi x}}+\frac{1}{\sqrt{2}\pi^{3/2}}\int_a^{\infty}\frac{e^{2x-s}}{(s-x)^{3/2}}K_0(s)ds. \tag{12}$$

The total Nusselt number is the mean value of $h_p(x)$ times the characteristic length $(a/2)$ of the problem:

$$Nu_p(a)=\frac{1}{2}\int_0^a e^x\rho(x)dx=\frac{\rho(k=-i)}{2}. \tag{13}$$

The integral in (11) for $k=-i$ has an analytical expression.[7] It gives

$$Nu_p(a)=\frac{ae^a[K_0(a)+K_1(a)]}{\pi}. \tag{14}$$

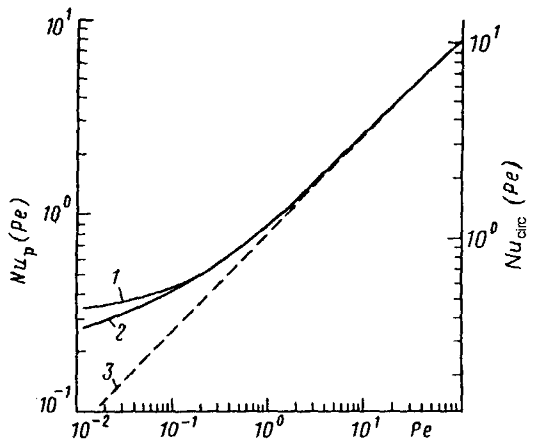

FIG. 1. The total Nusselt number for a flat plate and circular cylinder as a function of the Péclet number: *1*—Analytic expressions (14) and (18); *2*—numerical results from Ref. 9; *3*—asymptotic expression for very large Pe (15).

The plot of $Nu_p(a)$ is shown in Fig. 1. Equations (12) and (14) are the main results of our calculations.

In the case of very large a we can use the asymptotic expansion for $K_n(s)$ at large s.[7] It gives

$$Nu_p(s) \approx \sqrt{\frac{2a}{\pi}}. \tag{15}$$

Within this accuracy the local Nusselt number is

$$h_p(x) \approx \sqrt{\frac{2}{\pi x}} + \frac{e^{-2(a-x)}}{\pi\sqrt{a(a-x)}} + \sqrt{\frac{2}{a\pi}}\,\mathrm{erfc}(2\sqrt{a-x}). \tag{16}$$

The plot of $h_p(x)$ is shown in Fig. 2.

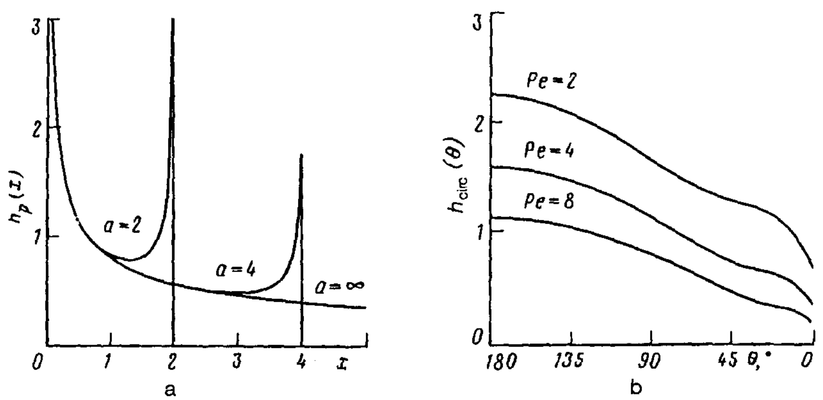

FIG. 2. The local Nusselt number for a plate (a) and circular cylinder (b) for some values of the Péclet number.

In contrast to $\rho_0(x)$, the function $\rho(x)=\rho_0^- + \rho_1^-(x)$ is equal to zero at $x>a$. However, this function is not zero at $x<0$. Equation (16) shows, however, that the contribution to the Nusselt number from the region $x<0$ is on the order of $\exp(-2a)$. This is the accuracy of our method. We also can use the Wiener–Hopf method repeatedly: cut the function $\rho_1(x)$ at $x<0$ and introduce a new function ρ_2 to compensate for this cutting at $x>0$, etc. The idea is clear, but each stage needs calculation of complicated sequences of the Fourier transformations or convolutions. A comparison with the results of numerical calculations[9] (see Fig. 1) shows that Eqs. (12)–(14) give an accuracy on the order of 10^{-2} at $a=0.3$, and the error rapidly (exponentially) vanishes with increasing a.

The Nusselt numbers for various cylinders can be obtained by a conformal transformation.[2] Let $z(\omega)$ be a conformal transformation which collapses the border of the cylinder C at the complex plane $\omega=\zeta+i\eta$ into a line of length L at the plane $x=x+iy$, and $z'(\omega)$ tends to unity at large $|\omega|$. The function $z(\omega)$ is the (scaled) complex potential of the flow with a velocity v_0 at infinity: $v_\zeta(\omega)-iv_\eta(\omega)=v_0 z'(\omega)$. The thermal conductivity equation is invariant under conformal transformation. The single parameter of the problem is the (dimensionless) length of the line: $a=Lv_0/(2\kappa)=Pe$ (the Péclet number). The local Nusselt number of the cylinder, h_c, is related to the solution for the plate h_p by

$$h_c(\omega)=h_p(x)|z'(\omega)|. \tag{17}$$

The total heat flux along the cylinder is a function of the Péclet number only. It gives the total Nusselt number. Here are some examples:

Circular cylinder with radius R_0, $\omega=Re^{i\theta}$. The complex transform is $z=\omega+R_0^2/\omega+2R_0$ and $L=4R_0$. It gives the Péclet number and the total Nusselt number (based on $2R_0$):

$$Pe=\frac{2R_0v_0}{\kappa} \quad \text{and} \quad Nu_{\text{circ}}(Pe)=\frac{4}{\pi}Nu_p(Pe). \tag{18}$$

The local Nusselt number is $h_{\text{circ}}(\theta)=2h_p(x)\sin\theta$, where $x=a(1+\cos\theta)/2$ (see Fig. 2b).

Elliptical cylinder with semiaxes b_x and b_y. Similar procedure yields the Péclet and total Nusselt numbers (based on b_x+b_y):

$$Pe=\frac{(b_x+b_y)v_0}{\kappa} \quad \text{and} \quad Nu_{\text{ell}}(Pe)=\frac{Nu_p(Pe)(b_x+B_y)}{P(b_x,b_y)}, \tag{19}$$

where $P(b_x,b_y)$ is the perimeter.

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