Asymptotically exactly solvable models of random walks and oscillations in disordered systems

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Some new, asymptotically exactly solvable models of random walks and oscillations in disordered systems are proposed. "Asymptotically" here means after a long time. Some of the new models are multidimensional. One-dimensional random walks with a nontrivial unit cell and 2D oscillations in a simple 2D lattice are examined in detail. A general method for constructing models of this sort is formulated. © *1994 American Institute of Physics.*

1. The theory of a random walk in a disordered medium is of general physical importance, because of its widespread applications in various fields of natural science and also because of its profound relationships with general problems in statistical physics and field theory (see, for example, some recent papers[1–8] and the literature cited there). The central problem in this theory is to construct $P_{xy}(t)=\langle\tilde{P}_{xy}(t)\rangle$, a solution of the kinetic equation

$$d\tilde{P}_{xy}(t)/dt=-\sum_{z}\left[\tilde{w}_{zx}\tilde{P}_{xy}(t)-\tilde{w}_{xz}\tilde{P}_{zy}(t)\right],\quad \tilde{P}_{xy}(0)=\delta_{xy}, \tag{1}$$

averaged over a random distribution of transition rates $\tilde{w}_{xz}$. Here $\tilde{P}_{xy}(t)$ is the conditional probability for observing an excitation at time t at lattice site **x** if this excitation was at **y** at time $t=0$.

Among the major accomplishments of this theory we can cite progress in research on the long-term asymptotic behavior of 1D systems which has been carried out since the pioneering studies.[9,10] Exact results on multidimensional problems are much fewer in number, and the only reliable calculations of diffusion coefficients for systems with rate fluctuations $\tilde{w}_{xz}$ which are not small are for the model of isotropic random hops[11] (the random-trap model) and modifications thereof.[6,8] Preasymptotic processes have also been analyzed in that model.[8]

The theory of oscillations in disordered media is close in many ways to the theory of a random walk in a disordered medium in cases in which the observables are the asymptotic behavior at long times, the asymptotic behavior at low propagator frequencies, or the low-energy behavior of the density of states.[1,12] Here the problem with random masses is an analog of the model of isotropic random hops. Asymptotically exactly solvable multidimensional models of oscillations in disordered media with random forces have apparently not been taken up previously in the literature.

In this letter we propose some new, asymptotically exactly solvable models. In the 1D case, these models are significantly richer than the existing models[1,6,8,9,11] in terms of the structure of the elementary transitions. In the multidimensional case we propose, for the first time, an asymptotically exactly solvable model of vector oscillations with random force matrices. An important point is that these new models are consistent with natural symmetry properties in the distribution of the parameters of the medium.

2. Let us consider symmetric random walks on the line of integers Z, partitioned into cells of length N. We denote by n the cell number, and by ν the number of the site in the cell. The nonvanishing $\tilde{w}_{xz}=\tilde{w}_{zx}$ are $\tilde{w}_{Nn+\nu-1,Nn+\nu}=\xi_n^\nu$, $\nu=1-N$, $\tilde{w}_{Nn-N,Nn}=\xi_n$. Introducing $\tilde{P}_{mn}^{\mu\nu}(t)=\tilde{P}_{Nm+\mu,Nn+\nu}(t)$, we find

$$d\tilde{P}/dt=-B\xi C\tilde{P},\quad \tilde{P}_{mn}^{\mu\nu}(t=0)=\delta_{\mu\nu}\delta_{mn},$$

$$B_{mn}^{\mu\nu}=(\delta_{\mu\nu}-\delta_{\mu+1,\nu})\delta_{mn}+\delta_{\mu N}\delta_{\nu 1}(\delta_{mn}-\delta_{m+1,n}),\quad C=B^+, \tag{2}$$

$$\xi_{mn}^{\mu\nu}=\delta_{mn}(\hat{\xi}_m)^{\mu\nu}=\delta_{mn}\xi_m^{\mu\nu}=\delta_{mn}(\xi_m^\mu\delta_{\mu\nu}+\xi_m),$$

where $\delta_{\mu+N,\nu}=\delta_{\mu\nu}$. We are assuming that the random matrices $\hat{\xi}_m$ have identical distributions which are independent of m, and we are assuming that there exists a reciprocal moment $\langle(\hat{\xi}_m)^{-2J}\rangle$, where $J>1$. Applying the technique of Ref. 8, which is completely legitimate for matrix-valued $\hat{\xi}_m$, we find an expansion of the Laplace transform of the propagator, $\tilde{P}(\lambda)=\int_0^{+\infty}dt\tilde{P}(t)\exp(-\lambda t)$:

$$\tilde{P}(\lambda)=(\lambda+B\xi C)^{-1}=\lambda^{-1}[1-B\xi(\lambda+CB\xi)^{-1}C],$$

$$\xi(\lambda+CB\xi)^{-1}=(\lambda/\xi+CB)^{-1}=\sum_{j=1}^{J}(G\lambda\eta)^jG+(G\lambda\eta)^{J+1}(\lambda/\xi+CB)^{-1}, \tag{3}$$

$$G=(\lambda/\kappa+CB)^{-1},\quad \eta=\kappa^{-1}-\xi^{-1},\quad \kappa^{-1}=\langle\xi^{-1}\rangle.$$

Taking the sum of (3) term by term, and expanding each term in an asymptotic series at small values of λ, we find the expansion of the average propagator which we need. In the leading order as $\lambda\to 0$, $\mathrm{Re}(\lambda)>0$, we have

$$\langle\tilde{P}(\lambda)\rangle\cong(\lambda+B\kappa C)^{-1}, \tag{4}$$

which involves a diffusive asymptotics at large t (Ref. 8).

3. Yet another simple and interesting 1D model can be found by the same approach, based on Eq. (2), by setting $B=1-T$, where $(Tf)_x=f_{x+1}$, and by assuming $\xi_{mn}^{\mu\nu}=\delta_{mn}\xi_m^{\mu\nu}$. The random matrices $\xi_m^{\mu\nu}$are limited only by the conditions that the transition rates $\tilde{w}_{xz}$ be nonnegative, that the first $2J$ reciprocal moments exist, and that the distributions of ξ_m in the different cells be translationally invariant and independent. In this case we find a richer system of relationships, in which all transitions within a group formed by a unit cell and the first site of thel nearest cell on the right are allowed.

4. The third system describes 2D oscillations in a disordered medium. We assume that "molecules" of unit mass are at the sites $x=(x_1,x_2)$ of a Z^2 lattice with an even sum of coordinates x_1+x_2, and we assume that each of them interacts with the two nearest coordination spheres from the same sublattice, in such a way that the displacements u_x^n satisfy the equations

$$d^2u/dt^2=-\Phi\xi\Phi^+\mu,\quad u(t=0)=v,\quad du(t=0)/dt=w,$$

$$u_x=(u_x^1,u_x^2),\quad \Phi_{xy}^{1n}=a_n\nabla_{xy}^1-\sigma b_n\nabla_{xy}^2,\quad \Phi_{xy}^{2n}=b_n\nabla_{xy}^1+\sigma a_n\nabla_{xy}^2,\quad \sigma^2=1, \tag{5}$$

$$\nabla_{xy}^j=\delta_{x+e_j,y}-\delta_{x-e_j,y},\quad e_1=(1,0),\quad e_2=(0,1),\quad (\xi)_{xy}=\xi_x\delta_{xy}.$$

Here $m,n=1–2$; a and b are real vectors; and the ξ_x are positive-definite, real, symmetric 2×2 matrices which are distributed identically and which are independent at different values of x. These matrices are defined at the sites with an odd sum of coordinates x_1+x_2. The force matrix $\Omega_{xz}^{mn}=(\Phi\xi\Phi^+)_{xz}^{mn}$ links each site x with the eight nearest sites of the same sublattice. Momentum is conserved: $\Sigma_x\Omega_{xz}^{mn}=0$. In leading order in the number ($N\to\infty$) of sites in the sublattice, and for an interaction of each site with the same nearest neighbors, the symmetric matrix Ω can have $17N$ independent elements. In our model, there are $3N$ such elements: the number of quantities ξ_x^{mn}.

It is easy to verify that the distribution $\Phi\xi\Phi^+$ is invariant under translations and rotations through $\pi/2$ of the plane of the lattice:

$$x\to sx,\quad u\to su\quad (se_1=e_2,\quad se_2=-e_1). \tag{6}$$

If the distributions ξ_x and $s\xi_x s^+$ are instead identical, then with $\sigma=1$, $a_1=a_2$, and $b_1=-b_2$ the distribution $\Phi\xi\Phi^+$ is invariant under reflections at the axes (in this case we have $ab=0$).

If averages $\langle(\xi_m)^{-2J}\rangle$, $J>1$, exist, the asymptotic form of average solution (5) is constructed by the same scheme as for (2), with λ replaced by λ^2, B by Φ, and C by Φ^+, and with initial conditions of a different form.

If the distribution ξ is invariant under (6), then $\langle\xi_x^{-1}\rangle=1/\kappa$ is a scalar matrix, and in leading order as $\lambda\to 0$, $\mathrm{Re}\lambda>0$ we have

$$\langle u(\lambda)\rangle\cong(\lambda^2-\kappa H)^{-1}(v+\lambda w),\quad H=H_1+H_2,$$

$$H_1^{mn}=b^2\Delta\,\delta_{mn}+(a^2-b^2)\nabla^m\nabla^n\sigma^{m+n},\quad \Delta=(\nabla^1)^2+(\nabla^2)^2,$$

$$H_2^{mn}=ab\{(1-\delta_{mn})[(\nabla^1)^2-(\nabla^2)^2]+2\sigma\nabla^1\nabla^2\delta_{mn}(-1)^m\}.$$

The operator H_2 is invariant under rotations (6), and H_1 is also invariant under reflections. In the continuum limit (replacing ∇^i by $\partial/\partial x_i$), with $\sigma=1$ and $ab=0$, the operator H is $O(2,R)$-invariant. In the low-frequency asymptotic limit, the spectrum has longitudinal and transverse acoustic branches $[\omega_\alpha=c_\alpha k(1-i(\zeta_\alpha k)^2),\ \alpha=l,t]$. The decay is governed entirely by the following term [proportional to $\langle n\lambda^2(\lambda^2/\kappa+\Phi^+\Phi)^{-1}\eta\rangle$] in the expansion of the effective force matrix H in fluctuations $\eta=\kappa^{-1}-\xi^{-1}$ [see Eq. (3)].

5. Systems (2) and (5) are examples from a very large class of asymptotically exactly solvable models, which take the following form in the case of a random walk in a disordered system:

$$d\tilde{P}_n^\mu/dt=-(B\xi^1F^1\xi^2...F^{M-1}\xi^MC\tilde{P})_n^\mu.$$

Here 1) $n\in Z^d$ is the "number" of the cell, and μ is the number of the site in the cell; 2) the operators CB, F^1, ..., F^{M-1} are invariant under displacements along n; 3) $((F^m)^{-1})_{nn'}^{\mu\mu'}=0$ for $|n-n'|>R_1$, $m=1–M-1$; 4) for $\mathrm{Re}\lambda>0$ the operator

$G=(\lambda+CB)^{-1}$ is l_1 -bounded, and we have $\lambda G_{nn'}^{\mu\mu'}=o(1)$ if $\lambda\to 0$ and $\mathrm{Re}\lambda>0$; 5) $(\xi^m)_{nn'}^{\mu\mu'}=(\xi^m)_n^{\mu\mu'}\delta_{nn'}$, $m=1-M$, where $(\xi^m)_n^{\mu\mu'}$ are distributed invariantly with respect to the same displacements, and $(\xi^m)_n^{\mu\mu'}$ and $(\xi^{m'})_{n'}^{\nu i'}$ are independent for $|n-n'|>R_2$; 6) the averages $\langle(\xi^1F^1\xi^2...\xi^M)^{-n}\rangle$, $n=1-2J$, where J is quite large, are finite; 7) the transition rates are nonnegative; and 8) $\Sigma_{n,\mu}(B\xi^1F^1\xi^2...F^{M-1}\xi^MC)_{nn'}^{\mu\mu'}=0$. In other words, the total probability is conserved. A randomness of the initial parameters is permissible. Here, nearly as in (3), an expansion in fluctuations of the operators $\lambda(\xi^1F^1\xi^2...\xi^M)^{-1}$ serves the purpose (a proof of the asymptotic convergence is basically the same as in Ref. 8).

In a similar way, we construct oscillation models in which a nontrivial unit cell is also permissible, and instead of 7) and 8) we require that the force matrix be symmetric, and that momentum be conserved.

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