Bosonization of a coordinate ring of $U_q[SL(N)]$: The cases of $N=2$ and $N=3$

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The non-Abelian coordinate ring of $U_q[SL(N)]$ (quantum deformation of the algebra of functions) for $N=2,3$ is represented in terms of conventional creation and annihilation operators. This allows us to construct explicitly the representations of this algebra, which were earlier described in somewhat more abstract algebraic fashion. Generalizations to $N>3$ and Kac–Moody algebras are not discussed but look straightforward. © *1994 American Institute of Physics.*

1. Free-field representations of Lie algebras

All algebras which are relevant for applications in theoretical physics usually have the "free-field representations" which allow one to express all the generators of the algebra in terms of the creation and annihilation operators, i.e., to embed the original algebra into the universal inclusion of the (several versions of the) Heisenberg algebra. These kinds of representation for the 1-loop (Kac–Moody, Virasoro, W-etc.) algebras[1–3] is the basis of the modern conformal field theory. Their analogs for the ordinary (0-loop) Lie algebras is also well known and widely used.

For the simplest case of $SL(2)$ this is the familiar representation

$$J^-=\frac{d}{dx},$$

$$J^0=x\,\frac{d}{dx}-j, \tag{1}$$

$$J^+=x^2\,\frac{d}{dx}-2jx,$$

and it can be considered as the "zero-mode" part of the Wakimoto representation[2,3] for $SL_{\hat{k}}(2)$,

$$\begin{aligned}
&J^-(z)=W(z),\\
&J^0(z)=\chi W(z)-q\partial\phi(z),\\
&J^+(z)=\chi^2W(z)-2q\chi\partial\phi(z)-k\partial\chi(z),\\
&q=\sqrt{k+2},\quad W(z)=\frac{\delta}{\delta\chi(z)},
\end{aligned} \tag{2}$$

which for $k=1$ becomes the standard Frenkel–Kac representation[1] for $SL_1(2)$,[4]

$$J^{\pm}(z)=C_{\pm}e^{\pm\Phi(z)},$$
$$J^{0}(z)=\partial\Phi(z). \tag{3}$$

The free-field representations should and can be generalized to include the quantum groups. The analog of 3 for $k=1$ and $q\neq 1$ is known as the Frenkel–Jing representation[5] (see also Ref. 6). Generalizations of (2) for any k,[21] though easy to derive, are explicitly available only in the nontransparent terms of the three scalar fields (instead of one scalar and β,γ system; see Ref. 7 for a brief review). The analog of (1) can be easily obtained in terms of the finite-difference operators, e.g.,

$$J_q^{-}=D^{+},$$
$$q^{\pm J_q^0}=q^{\mp j}M^{\pm}, \tag{4}$$
$$J_q^{+}=\frac{2q^{j+1}}{q+1}z^{2j+2}D^{+}z^{-2j}M^{-}.$$

Here

$$D^{\pm}F(x)=\frac{f(x)-f(q^{\pm 1}x)}{(1-q^{\pm 1})x},$$
$$M^{\pm}f(x)=f(q^{\pm 1}x), \tag{5}$$
$$M^{\pm}=I+(q^{\pm 1}-1)xD^{\pm}.$$

All these formulas (and their analogs for any N) can be easily deduced from the commutation relations for the generators T_α of any Lie algebra by the following procedure. We write

$$\mathcal{F}(\mathbf{x})=\langle\mathbf{j}|\prod_{\alpha>0}\hat{e}_q(x_\alpha T_{-\alpha})=\langle\mathbf{j}|\hat{e}_q\left(\sum_{\alpha>0}f_q^{\alpha}(\mathbf{x})T_{-\alpha}\right), \tag{6}$$

where α are arbitrarily ordered labels of all the generators, and the bra-vacuum $\langle\mathbf{j}|$ is annihilated by all the "positive" generators T_α, $\alpha>0$ and is the eigenvector of all the "Cartanian" (mutually commuting) eigenvectors, T_α, $\alpha\in\{0\}$, the eigenvalues are defined by the set $\mathbf{j}$. For nonquantum groups $q=1$ and the q-exponent[1)] $\hat{e}_q$ in (6) is replaced by the ordinary exponential function. The functions $f_1^{\alpha}(\mathbf{x})$ are polynomials in the $\mathbf{x}$-variables; the degree of the polynomial is equal to the number of items in the decomposition of α in the sum of the simple roots. The representation of the original algebra is now defined by the relation

$$J_\alpha\mathcal{F}(\mathbf{x})\doteq\mathcal{F}(\mathbf{x})T_\alpha, \tag{7}$$

where T_α at the r.h.s. is "carried" through the exponential operator to act on the vacuum, and terms arising from commutation of the operators can be imitated by taking x_β derivatives. We then have

$$J_\alpha J_\beta\mathcal{F}(\mathbf{x})=J_\alpha\mathcal{F}(\mathbf{x})T_\beta=\mathcal{F}(\mathbf{x})T_\alpha T_\beta. \tag{8}$$

It is thus easy to derive not only (1)–(4), but also all the other formulas from Ref. 3 and from other papers on free-field representations.

2. The coordinate ring of the quantum group

The purpose of this letter is to discuss the free-field representation of the new object: the coordinate ring of the quantum group, which is an essential piece of the theory but has a trivial classical limit since $q=1$, in which it becomes just a free Abelian algebra. For $q\neq 1$ this algebra is no longer Abelian and provides a solution of the basic equation[10] $\mathcal{R}(T\otimes T)=(T\otimes T)\mathcal{R}$, where $\mathcal{R}$ is the **R**-matrix, i.e., the solution of the Yang–Baxter equation.

In the case of $U_q(SL(2))T=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, where the elements a,b,c,d (the $q\neq 1$ analogs of the matrix elements) are no longer c-numbers, but operators with the commutation relations

$$\begin{aligned} &ab=qba,\\ &ac=qca,\\ &ad-da=(q-q^{-1})bc,\\ &bc=cb,\\ &bd=qdb,\\ &cd=qdc. \end{aligned} \tag{9}$$

This algebra can be considered as algebra of the linear automorphisms of the "quantum phase plane",[11] which is parametrized by the noncommuting "coordinates"

$$u_1u_2=qu_2u_1 \tag{10}$$

(they can be considered as exponentials of the coordinate and momentum operators, $u_1=e^Q$, $u_2=e^P$, $q=e^{ih}$, $P=-ih(d/dQ)$.

Commutation relations for the entries of T-matrices, which are associated with $U_q[SL(N)]$, can be easily described in terms of those for $U_q[SL(2)]$. If $T=(A_{ij})$, i, $j=1\ldots N$, then for any fixed $i<k$, $j<1$ the 2×2 matrix $\left(\begin{smallmatrix} A_{ij} & A_{il} \\ A_{kj} & A_{kl}\end{smallmatrix}\right)$ has exactly the same properties as $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in (9) [i.e., $A_{ij}A_{il}=qA_{il}A_{ij}$, $A_{ij}A_{kl}-A_{kl}A_{ij}=(q-q^{-1})A_{il}A_{jk}$, etc.].

Below we describe the representation of (9) and its analog for $U_q[SL(3)]$ in terms of the annihilation and creation operators. The case of arbitrary N and Kac–Moody algebras, as well as their relation to the more sophisticated representations of the quantum group [these are not the same, of course: compare (9) and (4)] will be discussed elsewhere. Instead we will discuss briefly the representations of the coordinate ring, which were found earlier in pure algebraic terms in Refs. 12 and 13 and can now be constructed explicitly in terms of the functions of commuting variables.

3. Oscillator representation of the basic algebra

Before addressing the question of bosonization of the algebra (9), we consider the even simpler "quantum hyperplane" algebras [compare with (10)]. These algebras will provide us with the building blocks for further constructions. We shall need two such algebras: the "chain" (or "quantum phase space") algebra,

$$\begin{aligned} &w_i w_j = q w_j w_i \quad \text{for } j=i+1, \\ &w_i w_j = w_j w_i \quad \text{for } |j-i|>1, \end{aligned} \tag{11}$$

and the "huperplane" algebra

$$u_i u_j = q u_j u_i \quad \text{for any } j>i. \tag{12}$$

Free-field representation (bosonization) expresses these generators in terms of those of Heisenberg algebra,

$$[\alpha_i, \alpha_j^\dagger] = \delta_{ij} \log q. \tag{13}$$

Such representation for (11) is straightforward:

$$w_i = e^{\alpha_{i-1}^\dagger + \alpha_i}. \tag{14}$$

The representation for (12) can be obtained from (14):

$$u_i = :\prod_{k\leq i} w_k: \;=\; :\exp\left(\sum_{k<i} \alpha_k^\dagger + \sum_{k\leq i} \alpha_k\right): . \tag{15}$$

All the operators involved are exponentials of the linear combinations of the creation and annihilation operators and normal ordering can be defined by just requesting that whenever the Wick theorem is applied for evaluation of correlation functions and/or commutation relations, no contractions of operators standing under the normal ordering signs should be included. For any such operator

$$\mathcal{O} = :\exp\left(\sum_k A_k \alpha_k + \sum_k B_k \alpha_k^\dagger\right): \tag{16}$$

we have

$$\mathcal{O}_1 \cdot \mathcal{O}_2 = \sqrt{\epsilon_{12}} :\mathcal{O}_1 \mathcal{O}_2: = \epsilon_{12} \mathcal{O}_2 \cdot \mathcal{O}_1, \tag{17}$$

where the c-number is $\epsilon_{12} = q \Sigma_k (A_k^{(1)} B_k^{(2)} - B_k^{(1)} A_k^{(2)})$. Since all the operators below will be of the form (16), in what follows we use (17) without a special reference.

4. The case of $U_q(SL(2))$

We can now describe bosonization formulas for the algebra (9). For this purpose we need two mutually commuting versions of algebra (12); their generators are denoted by $\{u_i\}$ and $\{v_i\}$, $u_i v_j = v_j u_i$. The commutation relations (9) are then immediately satisfied if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}. \tag{18}$$

The only relation which is slightly nontrivial is $ad-da=(q-q^{-1})bc$. It is here that (17) plays a crucial role. Indeed,

$$ad=:u_1v_1: \cdot :u_2v_2:=q:u_1v_1u_2v_2: ,$$
$$da=:u_2v_2: \cdot :u_1v_1:=q^{-1}:u_1v_1u_2v_2: , \tag{19}$$
$$bc=:u_2v_1: \cdot :u_1v_2:= :u_1v_1u_2v_2: =cb.$$

Representation (18) has an obvious generalization for $U_q(SL(N))$ with any N: it is sufficient to set

$$T=(A_{ij})=(u_iv_j). \tag{20}$$

However, both (18) and (20) are nongeneric representations: they are actually degenerate. This is clear, because the c-number $D=\det_qT=ad-qbc=da-q^{-1}bc$ in the case of (18) is identically vanishing: $D=0$. In the case of (20) the situation is even worse: both the full determinant of the matrix T and all its minors are identically vanishing. We now proceed to describe the generic nondegenerate representations.

For the case of $U_q[SL(2)]$ it is very simple to introduce the necessary correction: instead of (18) we can use

$$T=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2+D_2\dfrac{1}{u_1v_1} \end{pmatrix}. \tag{21}$$

Here $D_2=\det_qT$ is a c-number which commutes with all the operators u and v. Since the u's and v's are of the form (16), there are no problems with the definition of their negative powers. Using (12) along with its obvious corollary,

$$u_i\frac{1}{u_j}=q\frac{1}{u_j}u_i \quad \text{for } i>j, \tag{22}$$

it is easy to see that all the relations (9) are still applicable for representation (21).

5. The case of $U_q[SL(3)]$

The nondegenerate representation can now be written as follows:

$$T=\begin{pmatrix} a & b & e \\ c & d & f \\ g & h & k \end{pmatrix}=\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2\cdot W & u_2v_3\cdot W \\ u_3v_1 & u_3v_2\cdot W & u_3v_3\cdot W+D_3:\dfrac{u_4v_4}{u_5v_5}: \end{pmatrix};$$
$$W=1+\frac{1}{q}:\frac{u_3v_3}{u_1v_1u_2v_2u_4v_4}: \tag{23}$$

or

$$T=\begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2+:\dfrac{u_3v_3}{u_1v_1u_4v_4}: & u_2v_3+:\dfrac{u_3v_3^2}{u_1v_1v_2u_4v_4}: \\ u_3v_1 & u_3v_2+:\dfrac{u_3^2v_3}{u_1v_1u_2u_4v_4}: & u_3v_3+:\dfrac{u_3^2v_3^2}{u_1v_1u_2v_2u_4v_4}:+D_3:\dfrac{u_4v_4}{u_3v_3}: \end{pmatrix}. \tag{24}$$

The q-determinant of the entire matrix T is equal to $\det_q T=D_3$ and commutes with all the entries of T. The extra terms, which appear in the expressions for d,f,h, and k, are important for rendering the 2×2 minors of T nondegenerate; for example,

$$\Delta_{33}\equiv ad-qbc=:\frac{u_3v_3}{u_4v_4}: . \tag{25}$$

It is clear that Δ_{33} commutes with all the other contributions to the elements a,b,c,d of the $U_q[SL(2)]$ subalgebra. Representation of this subalgebra reduces to (21) if Δ_{33} is identified with D_2. However, in $U_q[SL(3)]$ Δ_{33} is no longer a c-number, but it still has simple (multiplicative-like) commutation relation with k:

$$\Delta_{33}k-q^2k\Delta_{33}=(1-q^2)D_3. \tag{26}$$

Such simple relations generally arise for the q-commutators of the minors for all the $U_q[SL(N)]$ algebras.[13]

The check that all the commutation relations of $U_q[SL(3)]$ are satisfied for representation (24) is a rather tedious but straightforward exercise. Again one should repeatedly make use of the relation (17). Let us check, for example, that $fk=qkf$. This is true since

$$\begin{aligned} fk&=\left(u_2v_3+:\frac{u_3v_3^2}{u_1v_1v_2u_4v_4}:\right)\times\left(u_3v_3+:\frac{u_3^2v_3^2}{u_1v_1u_2v_2u_4v_4}:+D_3:\frac{u_4v_4}{u_3v_3}:\right)\\ &=q^{1/2}:u_2u_3v_3^2:+(q^{3/2}+q^{-1/2}):\frac{u_3^2v_3^3}{u_1v_1v_2u_4v_4}:+q^{1/2}:\frac{u_3^3v_3^4}{u_1^2v_1^2u_2v_2^2u_4^2v_4^2}:\\ &\quad+q^{1/2}D_3\left(:\frac{u_2u_4v_4}{u_3}:+:\frac{v_3}{u_1v_1v_2}:\right), \end{aligned} \tag{27}$$

while

$$\begin{aligned} kf&=q^{-1/2}:u_2u_3v_3^2:+(q^{-3/2}+q^{1/2}):\frac{u_3^2v_3^3}{u_1v_1v_2u_4v_4}:+q^{-1/2}:\frac{u_3^3v_3^4}{u_1^2v_1^2u_2v_2^2u_4^2v_4^2}:\\ &\quad+q^{-1/2}D_3\left(:\frac{u_2u_4v_4}{u_3}:+:\frac{v_3}{u_1v_1v_2}:\right). \end{aligned} \tag{28}$$

The representation theory of the coordinate rings

According to Refs. 12 and 13, the representations of the $U_q[SL(N)]$ can be described in terms of the vectors obtained by the action of certain "creation operators"[2)] on the "vacuum." The "vacuum" is defined as the common eigenvector of the maximum set of commuting generators of the coordinate ring, while the "creation operators" are certain minors of the matrix T.[13] Realization of T in terms of the generators of the Heisenberg algebra allows us to construct all these objects explicitly. We give here only the example of $U_q[SL(2)]$.

The maximum set of commuting generators for generic q consists of b and c. Thus "vacuum" state is defined to satisfy

$$b|vac\rangle=\mu|vac\rangle, \quad c|vac\rangle=\nu|vac\rangle. \tag{29}$$

In order to obtain the highest weight representation we must set

$$d|vac\rangle=0, \tag{30}$$

and the entire representation consists of the vectors

$$a^n|vac\rangle. \tag{31}$$

For special values of $q=e^{2\pi i/p}$ with an integer p the set of commuting operators is actually bigger: d^P commutes with both b and c and, instead of (30), the "vacuum" can be defined as an eigenstate of d^P:

$$d^P|vac\rangle=\delta|vac\rangle, \quad \text{for } q^P=1, \quad p\in Z. \tag{32}$$

Using (21), we can now represent a,b,c, and d in terms of a pair of Heisenberg creation and annihilation operators:

$$\begin{aligned} &a=e^{\alpha+\beta}, \quad b=e^{\alpha^\dagger+\beta}, \quad c=e^{\beta^\dagger+\alpha}, \\ &d=e^{\alpha^\dagger+\beta^\dagger}+D_2e^{-\alpha-\beta}. \end{aligned} \tag{33}$$

We can now represent the Heisenberg generators in terms of the differential operators,

$$\begin{aligned} &\alpha^\dagger=x \log q, \quad \alpha=\frac{d}{dx}, \\ &\beta^\dagger=y \log q, \quad \beta=\frac{d}{dy}. \end{aligned} \tag{34}$$

We can then write a,b,c, and d in the form of the finite-difference operators

$$\begin{aligned} &a=m_x^+m_y^+, \quad b=q^ym_x^+, \quad c=q^xm_y^+, \quad d=q^{x+y}+D_2m_x^-m_y^-, \\ &m_x^\pm\equiv e^{d/dx}, \quad m_x^\pm f(x)=f(x\pm 1). \end{aligned} \tag{35}$$

Therefore, as long as we deal with the coordinate ring only, and not with the Heisenberg algebra itself, x and y can be considered as variables on the integer lattice: $x,y\in Z$. The solution of Eqs. (29) is now given by

$$|vac\rangle\sim q^{-xy}\mu^x\nu^y\equiv|\mu,\nu\rangle. \tag{36}$$

Condition (30) or (32) can be now considered as definitions of D_2 in (35) in terms of μ, ν, and δ: condition (30) implies that

$$D_2=-q\mu\nu, \tag{37}$$

while condition (32) means that

$$\delta = \prod_{k=1}^{p} \left(1 + \frac{D_2}{q^{2k+1}\mu\nu} \right). \tag{38}$$

The representation itself consists of the states

$$|n\gg \equiv a^n|vac\rangle \sim q^{-(x+n)(y+n)}\mu^{x+n}\nu^{y+n} = q^{-n^2}\mu^n\nu^n|q^{-n}\mu, q^{-n}\nu\rangle. \tag{39}$$

For $q=e^{2\pi i/p}$, there are at most p linearly independent states in this representation. For even p, there are actually irreducible representations of the size $p/2$. This is clear from the fact that all the states (39) are eigenstates of b and c, and that the action of the operators a and d in (35) does not change parity of the integer-valued combination $x+y$. The representation (39) can therefore be defined on the sublattice $x,y \in Z$, $x+y \in 2Z$ and $|-\mu,-\nu\rangle$ can be identified with $|\mu,\nu\rangle$.

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[1)]We recall that the q-exponent $e_q(x)=1/E_q(-x)$ is characterized by the following set of properties:

1. $D^+\hat{e}_q(x)=\hat{e}_q(x)$; $\hat{e}_q(x)\equiv e_q((1-q)x)$ and $\lim_{q\to 1}\hat{e}_q(x)=e^x$;
2. $e_q(x)=\Sigma_{k\geq 0}x^k/(q,q)_k$, $E_q(x)=\Sigma_{k\geq 0}q^{k(k-1)/2}x^k/(q,q)_k$, where $(a,q)_k\equiv\Pi_{i=0}^{k-1}(1-aq^i)$ $=(a,q)_\infty/(aq^k,q)_\infty$;
3. $e_q(x)=1/(x,q)_\infty$. Consequently, $E_q(x)=(-x,q)_\infty$ and $\theta_{00}(x)\equiv\Sigma_{k=-\infty}^{\infty}q^{k^2/2}x^k$ $=(q,q)_\infty E_q(q^{1/2}x)E_q(q^{1/2}x^{-1})$;
4. $E_q(x)E_q(y)=E_q(x+y)$ and $e_q(y)e_q(x)=e_q(x+y)$, provided that $xy=qyx$;
5. $E_q(y)E_q(x)=E_q(x+y+yx)$ and $e_q(x)e_q(y)=e_q(x+y-yx)$, provided that $xy=qyx$. This identity was obtained by Faddeev and Volkov.[9]

The first three properties explain the relevance of the q-special functions as solutions to finite-difference equations (i.e., to various periodicity constraints), while the last two properties are crucial for the occurrence of the same functions in the study of noncommutative algebras and problems of quantum mechanics and quantum field theory.

[2)]Not to be confused with the Heisenberg operators $\alpha^\dagger$ and $\beta^\dagger$.

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