Stochastic properties of multidimensional cosmological models near a singular point

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There is a wide class of multidimensional cosmological models, other than the Bianchi IX model (a mixmaster universe), which have the property of being stochastic. A criterion for the occurrence of a stochastic behavior is proposed for various models. © *1994 American Institute of Physics.*

1. The homogeneous cosmological model Bianchi IX exhibits a random behavior near a singularity.[1] This model is important for cosmology, primarily because its behavior is a prototype for the behavior of a general solution of Einstein's equations near a singular point:[2] Specifically, near a singularity, the action for an arbitrary gravitational field breaks up into a sum of independent terms, each of which is the action of a homogeneous model, as was shown in Ref. 3. The random nature of the dynamics leads to the excitation of inhomogeneities of progressively larger coordinate scales, and it ultimately determines the statistical properties of these inhomogeneities.[3,4]

In addition, there is obvious interest in the possibility of generalizing these results to the case of a multidimensional cosmology.[5,6] Hints at the existence of such a cosmology follow from various unification theories,[7] for which the standard Einstein theory of gravitation is merely a low-energy limit. Obviously, additional dimensions (if they exist) should be manifested most clearly under extreme conditions—near a singular point.

The first indication of the spontaneous onset of a stochastic nature in a multidimensional model was found in Ref. 8. It was shown there that in the case $D=5$ (D being the dimensionality of the space–time) there is an oscillatory approach to the singularity, as in Bianchi IX. We might add that the question of the transition of the behavior of various multidimensional models to a stochastic nature has been studied by a variety of methods in several places.[9] We show below that a random behavior near a singular point is a general property of a wide class of multidimensional cosmological models.

2. We consider a homogeneous cosmological model with a spatial dimensionality $n+1$. The metric of such a model is

$$ds^2=N^2dt^2-\sum_{a=0}^{n} e^{q^a}\sigma_\alpha^a\sigma_\beta^a dx^\alpha dx^\beta, \tag{1}$$

where $\sigma^a=\sigma^a_\alpha dx^\alpha$ are given homogeneous basis forms which obey equations with the structure

$$d\sigma^a=\frac{1}{2}C^a_{bc}\sigma^b\wedge\sigma^c, \tag{2}$$

where C^a_{bc} are structure constants of some semisimple Lie algebra. The action for such a model is, in Planckian units,

$$I=\int\left\{p_a\frac{dz^a}{dt}-\lambda C(p,z)\right\}dt, \tag{3}$$

where the variables z^a are related to the scale functions by the linear transformation $z^a=A^a_bq^b$ with the constant coefficients[5] A^a_b λ is a Lagrange multiplier related to the evolution function N by $\lambda=N\exp(-(1/2)\Sigma q^a)$, and the coupling Hamiltonian is

$$C=\eta^{ab}p_ap_b+V(z). \tag{4}$$

Here $\eta^{ab}=\mathrm{diag}(-1,1,\ldots1)$, is the metric of the minisuperspace M^{n+1}, and the potential in (4) has the structure

$$V(z)=\sum_{i=1}^{m}r_i\exp(u^i_az^a). \tag{5}$$

The quantities r_i and u^i_a are constants, which are to be determined from the structure constants C^a_{bc} and the matrix A^a_b Cosmological models with a multicomponent ideal fluid[6] lead to similar potentials (here r^i and u^i_a are arbitrary).

We consider the following system of superspace coordinates (analogs of the Misner–Chitre coordinates[10,11]):

$$z^0=-e^{-r}\frac{1+y^2}{1-y^2},\quad \mathbf{z}=-2e^{-r}\frac{\mathbf{y}}{1-y^2},\quad y=|\mathbf{y}|<1. \tag{6}$$

This coordinate system limits dynamic system (3) to the lower "light" cone $\nu_-=\{z=(z^0,\mathbf{z})|z^0<-|\mathbf{z}|\}$. In the new coordinates, action (3) becomes

$$I=\int\left\{\pi_k\frac{dy^k}{dt}-h\frac{d\tau}{dt}-\bar{\lambda}\{\epsilon^2(y,\pi)+U(y,\tau)-h^2\}\right\}dt, \tag{7}$$

where $U=e^{-2r}V$, $\epsilon^2=1/4\,(1-y^2)^2\pi^2$, and $\bar{\lambda}=\lambda e^{2r}$. Solving the Hamiltonian-coupling equation $C=0$ for h, we can put action (7) in a formulation equivalent to the ADM formulation:

$$I=\int\left\{\pi_k\frac{dy^k}{d\tau}-h(\pi,\tau,y)\right\}d\tau, \tag{8}$$

where τ plays the role of the time (this situation corresponds to the choice of gauge $\bar{\lambda}=1/2h$), and the quantity $h=\sqrt{\epsilon^2+U}$ plays the role of the ADM Hamiltonian.

The reduced configuration space (i.e., the "spatial" part of the minisuperspace M^{n+1}) is the n-dimensional sphere $D^n=\{\mathbf{y} \in R^n \;\| \mathbf{y}|<1\}$. This sphere, along with the metric $\gamma_{kl}=4\delta_{kl}/(1-y^2)^2$, which is set by the kinetic term ϵ^2 in h, is one realization of the n-dimensional Lobachevskiĭ space H^n.

3. We now consider the asymptotic case $\tau\to-\infty$, which corresponds, for $z\in\nu_-$, to $z^0\to-\infty$, i.e., to a singular point in metric (1). In this asymptotic case, each component of U [see(5)], satisfying the conditions

$$(u^i)^2=(\mathbf{u}^i)^2-(u_0^i)^2>0\to r_i>0, \quad u_0^i>0, \tag{9}$$

has the form of a potential wall:

$$r_i\exp\{-2\tau+u_a^i z^a(y,\tau)\}\to\theta_\infty[A_i(y)], \tag{10}$$

where

$$\theta_\infty[x]=\begin{cases} +\infty, & x>0,\\ 0, & x<0,\end{cases} \tag{11}$$

and where the expression

$$A_i(y)=-\left(\mathbf{y}+\frac{\mathbf{u}^i}{u_0^i}\right)^2+\left(\frac{\mathbf{u}^i}{u_0^i}\right)^2-1=0 \tag{12}$$

specifies the position of the wall.

Conditions (9) are satisfied for a wide class of models with an ideal fluid[6] and also for several homogeneous models. We denote by Δ_+ the set of all terms of sum (5) in potential U which satisfy condition (9). The asymptotic expression for the potential then takes the form

$$V_\infty(y)=\sum_{i\in\Delta_+}\theta_\infty[A_i(y)]. \tag{13}$$

Dynamic system (8) near a singular point thus reduces to a billiard in Lobachevskiĭ space H^n. The boundary of the billiard is formed by a system of spheres D_i^n, $i \in \Delta_+$ [see (12)]. Serving as a criterion for a random nature of the billiard is the condition that its volume be finite.[12] In this case, the invariant measure to which an arbitrary initial distribution relaxes is given by

$$d\mu(y,s)=\mathrm{const}\frac{d^n y d^{n-1}s}{(1-y^2)^n},$$

where $\mathbf{s}$ is a unit velocity vector.

It turns out that the condition that the volume of the billiard be finite can be formulated in terms of the problem of the illumination of a sphere.[13] In other words, if point sources in R^n at the points $\mathbf{V}^i=-\mathbf{u}^i/u_0^i$ completely illuminate a sphere S^{n-1} of unit radius centered at the origin of coordinates, then the volume of the billiard is finite. The billiard is then random and has the property of mixing[14] (the Lyapunov exponent is found as the square root of the modulus of the curvature of the Lobachevskiĭ space). By virtue of conditions (9) we have $|\mathbf{V}^i|>1$; i.e., the point sources lie outside the illuminated

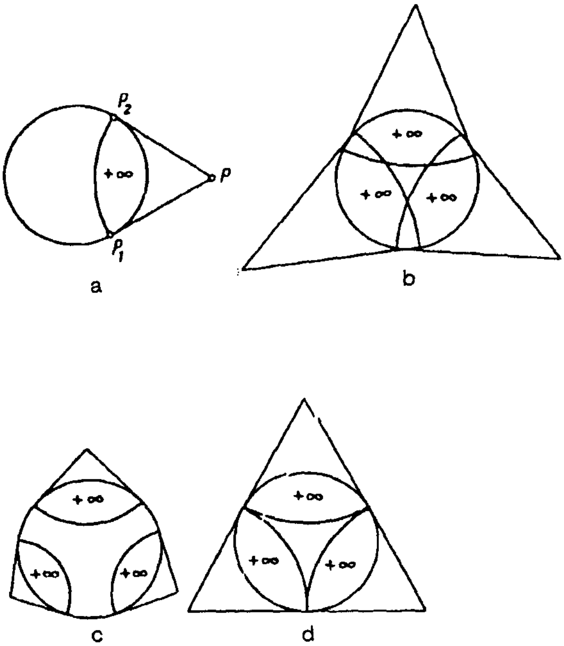

FIG. 1. Various billiard configurations in a Lobachevskiĭ plane ($n=2$). a) $m_+=1$; b) $m_+=3$, compact billiard; c) $m_+=3$, billiard of infinite volume; d) $m_+=3$, noncompact billiard with a finite volume (Bianchi IX).

sphere, and the minimum number of sources required to completely illuminate the sphere is equal to the dimensionality of the minisuperspace (i.e., $n+1$; Ref. 13).

Let us look at some simple examples. We denote by m_+ the number of spheres (12) which form the boundary of the billiard (to simplify the illustration, we assume $n=2$ everywhere). We set $m_+=1$. The billiard shown in Fig. 1a then exists. Its boundary (a potential wall) is formed by a circular arc which is centered at the point $P=\mathbf{V}=-\mathbf{u}/u_0$ and which has a radius $P_2P=(V^2-1)^{1/2}$. The points of the absolute, $|\mathbf{y}|=1$, which are inaccessible to trajectories of the dynamic system, are those that lie on the arc P_1P_2, illuminated by point P.

We now set $m_+>1$. The situations shown in Fig. 1, b, c, d (for $m_+=3$), are then possible. In Fig. 1, b and d, the billiard has a finite volume (although the billiard is not an intrinsic billiard in the latter case; i.e., points lying on the absolute are accessible to the trajectories). In Fig. 1c, the volume is infinite, so the billiard is not a mixing billiard.

For cosmological model (1), a random behavior near the singularity is thus possible only if the number of exponentials in the potential with a positive square of the vectors u_a^i [see (9)] is no lower than the dimensionality of the minisuperspace. Corresponding to the Bianchi IX model are the values $n=2$, $m=6$, $m_+=3$ (the three arguments in the exponential function have a null square and do not give rise to potential walls). The billiard has the form shown in Fig. 1d (Ref. 10). The billiard has a finite volume (i.e., it is a mixing billiard). The trajectories are geodesic lines of the Lobachevskiĭ plane. Corresponding to motion along a geodesic is a so-called Kasner regime of the evolution of the metric.[1] Reflection from the wall of the billiard corresponds to a change in Kasner regime (a change in Kasner epoch).

The quantum dynamics of these models can be described as in Ref. 15 for the Bianchi IX model.

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