Fractal dimension of turbulence

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(Submitted 27 April 1994; resubmitted 20 June 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **60**, No. 4, 247–251 (25 August 1994)

The experimental behavior of the fractal dimension of turbulence as a function of the Reynolds number of the flow, R, is explained theoretically. The exponent of the first-order structure function is compared with experiment over a broad range of R. © *1994 American Institute of Physics.*

Spatial intermittency is a fundamental property of small-scale turbulence.[1] Several theoretical models have been proposed for intermittency (see, for example, a review[2]) for developed turbulent flows (with $R \gg 1$). The experimental results of Refs. 3 and 4 indicate that even at relatively small values of R the characteristics of the small-scale turbulence exhibit a scaling with R-dependent exponents. In this letter we offer a theoretical explanation of the experimental results of Refs. 3 and 4 on the basis of a phenomenological model.[5–7]

The following equation can be derived for the moment function of the energy-dissipation rate ϵin equilibrium turbulence on the basis of a semiempirical model:[5]

$$\Phi'' + 2(\lambda_1 + \lambda_2 q)\Phi' + \lambda_3 q(q-1) = 0, \tag{1}$$

where $\Phi(q,x) = \langle \epsilon^q \rangle / \langle \epsilon \rangle^q$, the prime means differentiation with respect to x, x is the logarithm of the turbulent Reynolds number $\mathrm{Re} = k^2/(\langle \epsilon \rangle \nu)$, k is the average energy of the turbulence, and ν is the viscosity.

A more rigorous derivation of Eq. (1) was given in Ref. 7. It was shown there that a corresponding equation can be derived by separating "fast" and "slow" variables in the kinetic equation for the distribution of the dissipation in an arbitrary, generally nonequilibrium turbulence. In the latter case, x is the logarithm of an effective Reynolds number.

From the hypothesis of the independence of successive breakdowns, one finds the following relation for the breakdown coefficient of the dissipation field, $e_{r,l} = \epsilon_r / \epsilon_l$ (Ref. 8):

$$\langle (e_{r,l})^q \rangle = (l/r)^{\mu_q} \quad \text{for} \quad \eta \ll r, \ \ l \ll L. \tag{2}$$

Here ϵ_r is the dissipation averaged over a region with length scale r, and $\eta = \langle \epsilon \rangle^{-1/4} \nu^{3/4}$ and $L = k^{3/2}/\langle \epsilon \rangle$ are respectively inner and outer scales of the turbulence. If the distribution of $e_{r,l}$ is independent of both L and η in the scaling interval, then μ_q is a universal function of q. In the known intermittency models,[2] this hypothesis is based on the analysis of asymptotically large Reynolds numbers. A more common assumption is that, for independent breakdowns, the distribution of $e_{r,l}$ may depend on the ratio of the length scales η and L, i.e., on Re.

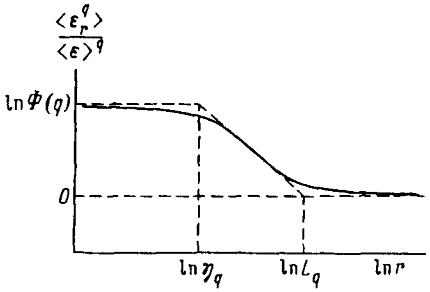

FIG. 1. Probabilistic moment of the partially averaged dissipation ϵ_r versus the length scale r.

If the exponents μ_q are independent of Re, the coefficients λ_1, λ_2, and λ_3 are universal constants.[6] The converse is generally not true. Let us assume that λ_1, λ_2, and λ_3 are constants. A general solution of Eq. (1) is then

$$\Phi(q,x)=C_1(q)\exp\{[-(\lambda_1+\lambda_2 q)-\sqrt{(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)}](x-x_0)\}$$
$$+C_2(q)\exp\{[-(\lambda_1+\lambda_2 q)+\sqrt{(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)}](x-x_0)\}, \qquad (3)$$

where the parameter x_0 has been introduced to simplify the equations that follow below.

Over the scaling interval we have $\langle\epsilon_r^q\rangle/\langle\epsilon\rangle^q=\kappa r^{\mu_q}$, where κ and μ_q are independent of r. In the limit $r\to\infty$ we evidently have $\langle\epsilon_r^q\rangle/\langle\epsilon\rangle^q\to 1$, while as $r\to 0$, we have $\langle\epsilon_r^q\rangle/\langle\epsilon\rangle^q\to\Phi(q,x)$. Figure 1 is a sketch of the moment of order q $(q>1)$ versus r. Let us determine L_q and η_q for a fixed q, working from the relations $\kappa(L_q)^{-\mu_q}=1$, and $\kappa(\eta_q)^{-\mu_q}=\Phi(q,x)$. We find

$$(L_q/\eta_q)^{\mu_q}=\Phi(q,x). \qquad (4)$$

The parameters L_q and η_q are equal in order of magnitude to the length scales L and η, respectively. Setting $L_q=a(q)L$ and $\eta_q=b(q)\eta$, where $a(q)$ and $b(q)$ are universal functions of q, we find from (4)

$$\left[g(q)\exp\left(\frac{3}{4}x\right)\right]^{\mu_q}=\Phi(q,x), \qquad (5)$$

where $g(q)=a(q)/b(q)$.

If the breakdowns are independent for any ratio l/r, the distribution of the logarithm of $e_{r,l}$ is infinitely divisible;[9] i.e., the distribution of a quantity can be represented as the sum of an arbitrary number of independent, identically distributed terms. If, for some q, the coefficient $C_2(q)$ is not zero, then the leading term of solution (3) in the limit $x\to\infty$ is the second term. Working from the properties of the Laplace transform of infinitely divisible distributions,[9] one can show[6] that this solution does not cause the distribution to be nonnegative. We thus have $C_2(q)\equiv 0$. Substituting (3) into Eq. (5), and omitting the index on the coefficient $C_1(q)$, we find

$$\mu_q=\frac{\ln[C(q)]+[-(\lambda_1+\lambda_2 q)-\sqrt{(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)}](x-x_0)}{\ln[g(q)]+\frac{3}{4}x}. \qquad (6)$$

In the limit $x\to\infty$ we find

$$\mu_q \to \frac{4}{3}[-(\lambda_1+\lambda_2 q)-\sqrt{(\lambda_1+\lambda_2 q)^2-\lambda_3 q(q-1)}],$$

and the distribution of the dissipation breakdown coefficient approaches the universal distribution found in Ref. 6. If $\lambda_2^2 \neq \lambda_3$, the limiting distribution must contain a δ-function,[6] but this conclusion contradicts experiment.[10] If the limiting distribution is not to vanish at $0<e_{r,l}<l/r$, in accordance with the experiment of Ref. 10, we must require[6] $\lambda_2=-3/4$. We thus find $\lambda_3=9/16$.

In the experiments of Refs. 3 and 4 it was found that the dimension of the turbulence varies from 2 at a certain critical value $R_{\rm cr}$ of the Reynolds number to nearly 3 at large values of R. The model of Refs. 5 and 7, like the classical theory,[11] was proposed for developed turbulence. We assume that Eq. (1) also holds at small values of R, close to $R_{\rm cr}$.

The spectrum of generalized dimensions D_q is found from the formula[12] $D_q=3+\mu_q/(1-q)$. If the dissipation is concentrated on a uniform fractal with a dimension 2 at $R_{\rm cr}$, then we have $D_q=2$ for arbitrary $q>0$, i.e., $\mu_q=q-1$. We then have $\ln[C(q)]=(q-1)[\ln(g(q))+\frac{3}{4}x_0]$, where x_0 is the logarithm of the critical Reynolds number. Hence

$$\mu_q=q-1+\frac{-4\lambda_1+3-\sqrt{(4\lambda_1-3q)^2-9q(q-1)}}{4\ln[g(q)]+3x}(x-x_0). \tag{7}$$

From the normalization condition $\mu_0=0$ we find $\ln[g(0)]=-(3/4)\,x_0$.

Denoting by δu_r the difference between the velocities at points at a distance r (r belongs to the scaling interval), we have

$$\langle \delta u_r \rangle \propto r^{\zeta_1} \propto r^h r^{(3-D_{1/3})}, \tag{8}$$

where h is the scaling exponent of the first-order structure function for the "active" turbulent region. The measurement method used in the experiments of Ref. 3 makes it possible to determine h directly. The dimension of the interface, found in Ref. 4, was interpreted in terms of the exponent h in Ref. 13. An explanation of the experiments of Ref. 3 based on a uniform β model[14] with an R-dependent fractal dimension D was proposed in Ref. 15. For the β model we have $h=(D-2)/3$. In the general case of a multiscaling dependence of the dissipation moments we would have $h=(D_{1/3}-2)/3$. Setting $\gamma=\ln[g(1/3)/g(0)]$, we find from (7)

$$h=\frac{-4\lambda_1+3-\sqrt{(4\lambda_1-1)^2+2}}{8\gamma+6(x-x_0)}(x-x_0). \tag{9}$$

Since the turbulent Reynolds number Re is proportional to R, we have

$$h=\frac{-4\lambda_1+3-\sqrt{(4\lambda_1-1)^2+2}}{8\gamma+6\ln(R/R_{\rm cr})}\ln(R/R_{\rm cr}). \tag{10}$$

Figure 2 shows experimental values of h for flow in a tube ($R_{\rm cr}=2160$), behind a grill ($R_{\rm cr}=263$) (Ref. 3), and in the wake behind a cylinder ($r_{\rm cr}=165$) (Ref. 4); also shown here is a theoretical plot from Eq. (10). For the parameter λ_1 we adopted the value -3.5 found in Ref. 6. That value leads to a satisfactory approximation of the distribution

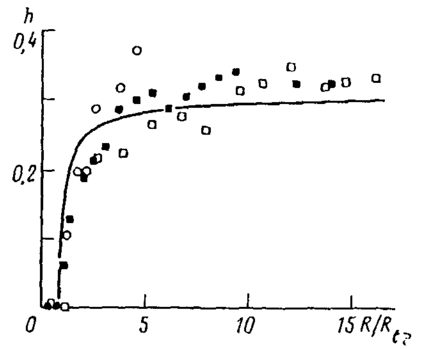

FIG. 2. Exponent of the first-order structure function versus the Reynolds number. Open and filled squares—Flow in a tube and behind a grill;[3] circles—wake behind a cylinder;[4] solid curves—calculated from Eq. (10). $\lambda_1=-3.5$, $\gamma=0.15$.

of the dissipation breakup coefficient in an independent experiment.[10] A variation of the one adjustable parameter γ yields $\gamma=0.15$ as the best value (according to visual inspection). The fair agreement between the theoretical curve and the experimental data is evidence that the model of Refs. 5–7 can also be used at relatively small values of the Reynolds number of the flow.

We studied the behavior of the scaling exponents of the dissipation moments as a function of the Reynolds number. The universal model of the distribution of the breakup coefficient[6] leads to overestimates of the exponents ζ_n of the higher-order structure functions.[7] The exponents ζ_n found in various experiments (both physical and numerical) are systematically at odds with each other. According to the free-hand plot in Ref. 16, it seems more reasonable to conclude that there is a family of $\zeta_n(R)$ curves rather than a universal dependence of ζ_n on n. It will be shown in a future paper that the arguments of the structure functions found on the basis of this model agree well with experiment.

One can show that, even without the hypothesis that the coefficients $\lambda_{1,2,3}$ are constant, that only one of the two fundamental solutions of Eq. (1) is of physical interest. To reach agreement with the experiments of Ref. 10 we need to set $\lambda_2=-3/4$ and $\lambda_3=9/16$, while the coefficient λ_1 may depend on the Reynolds number. If we have $\lambda_1\rightarrow\infty$ as $\mathrm{Re}\rightarrow\infty$, the dissipation distribution in the inertial interval asymptotically approaches that predicted by the Kolmogorov theory[11] with $h=1/3$. Whether the dimension of the turbulence has an asymptotic value of 3 or a value close to but different from 3 is a question of considerable theoretical interest.

Baudet *et al.*[17] have cast doubt on the interpretation of the measurements of Ref. 3. They found that the exponents ζ_n are universal over a broad range of Reynolds numbers, and that they cannot be determined at small values of R, because of the absence of an inertial interval. It was asserted that the R dependence of h (Ref. 3) stems from an incorrect choice of the boundaries of the inertial interval. One can offer the following explanation of the discrepancy between the results of Refs. 3 and 4, on the one hand, and those of Ref. 16, on the other. It can be seen from Fig. 1 that, for any finite R, scaling is an approximation, and a "linear" region (in logarithmic coordinates) is a segment of a tangent to the curve at the inflection point. With decreasing R, the range of scales which can be approximated with a given accuracy by a tangent decreases, but the inclination angle of the tangent, which specifies μ_q, can be determined even at small values of R.

A weak logarithmic $h(R)$ dependence at large R may be concealed by measurement errors.

We wish to stress that the validity of applying ideas from the theory of a locally isotropic turbulence to flows with relatively small values of R is under debate. It requires further, primarily experimental, study.

This work had partial financial support from the ISF (Grant MEV000).

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Translated by D. Parsons