Spectrum of a charged Bose gas in a quantizing magnetic field

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A charged Bose gas in a quantizing magnetic field below the critical temperature for Bose condensation, $T_{c2}(H)$, is analyzed. The Bose condensate forms a vortex lattice in this case. In a periodic potential of this sort, the transverse motion of a particle becomes unbounded because of the translational invariance of the lattice. As a result, the spectrum of excitations is of a band nature and is governed by the 2D quasimomentum in the perpendicular plane. The Green's function of the gas is found; the periodic vortex structure of the Bose condensate is taken into account. The spectrum of excitations of a Bose gas is derived for a square vortex lattice. © *1994 American Institute of Physics.*

The discovery of superconductivity in metal oxides has recently revived interest in the theory of a charged Bose gas. Several of the theories which have been offered to describe the properties of the metal oxides are based on the idea of local pairs. We will not go into the details of the mechanism by which local pairs form in these compounds, but we would like to mention the bipolaron theory,[1] in which a strong electron–phonon interaction gives rise to small-radius bipolarons. In the limit of a low particle density, a system of local pairs is a charged Bose gas, as was shown in Ref. 2.

The lower critical magnetic field H_{c1} and the thermodynamic field H_c were calculated in Ref. 3. It was shown that in intermediate magnetic fields $(H_{c1} \leqslant H \leqslant H_{c2})$ a charged Bose gas is in a mixed state. As was pointed out in Ref. 2 for a charged Bose gas, an external magnetic field quantizes the transverse motion of the bosons. At magnetic fields $H \leqslant H_{c2}$ a macroscopic number of particles accumulate in the lower energy level, which corresponds to the zeroth Landau level and to a zero momentum projection onto the magnetic field. In this case the situation is dominated by bosons in a single Landau level (the lower one; this is the ultraquantum limit). In a charged Fermi gas, Landau quantization can again play an important role in shaping the superconducting state.[4]

Our purpose in this letter is to derive the spectrum of excitations of a charged Bose gas in the mixed state. In fields $H \sim H_{c2} \gg H_{c1}$ a Bose condensate forms a vortex lattice, corresponding to an integer number of magnetic-flux quanta M. In this case, each Landau level splits into M subbands.[5] To simplify the interpretation of the results, we consider a model of the charged Bose gas with a short-range interaction potential. This model follows from the theory of small-radius bipolarons in the limit of a low carrier density.

To find the spectrum of excitations of a charged Bose gas in a magnetic field below the critical temperature T_{c2}, we generalize the diagram technique of Ref. 6 to the case of a nonuniform Bose condensate. In fields $H \sim H_{c2} \gg H_{c1}$, the condensate wave function $\xi_0(\mathbf{r})$ is sought in the form

$$\xi_0(\mathbf{r}) = \exp\left(i \frac{xy}{2l^2} \right) \sum_m C_m \exp\left(-i \frac{m a_x y}{M l^2} \right) \exp\left(- \frac{(x - m a_x / M)^2}{2l^2} \right), \tag{1}$$

where $C_{m+M} = C_m$ and $C_m = C_0$ for a square lattice, and $C_m = C_0 e^{i\theta m}$ and $\theta_m = \pi m^2/2$ for a triangular lattice.

The Green's function of bosons in a magnetic field is governed by the system of equations

$$G(\mathbf{r},\mathbf{r}';i\omega_n) = G^0(\mathbf{r},\mathbf{r}';i\omega_n) + \int d\mathbf{r}_1 d\mathbf{r}_2 G^0(\mathbf{r}_1,\mathbf{r}';i\omega_n) \Sigma^{11}(\mathbf{r}_2,\mathbf{r}_1;i\omega_n) G(\mathbf{r},\mathbf{r}_2;i\omega_n)$$

$$+ \int d\mathbf{r}_1 d\mathbf{r}_2 G^0(\mathbf{r}_1,\mathbf{r}';i\omega_n) \Sigma^{20}(\mathbf{r}_2,\mathbf{r}_1;i\omega_n) F(\mathbf{r},\mathbf{r}_2;i\omega_n), \tag{2}$$

$$F(\mathbf{r},\mathbf{r}';i\omega_{\_n}) = \int d\mathbf{r}_1 d\mathbf{r}_2 G^0(\mathbf{r}',\mathbf{r}_1;i\omega_n) \Sigma^{11}(\mathbf{r}_1,\mathbf{r}_2;i\omega_n) F(\mathbf{r},\mathbf{r}_2;i\omega_n)$$

$$+ \int d\mathbf{r}_1 d\mathbf{r}_2 G^0(\mathbf{r}',\mathbf{r}_1;i\omega_n) \Sigma^{02}(\mathbf{r},\mathbf{r}_2;i\omega_n) G(\mathbf{r}_1,\mathbf{r}_2;i\omega_n), \tag{3}$$

where $G^0(\mathbf{r},\mathbf{r}';i\omega_n)$ is found from the equation

$$\left[i\omega_n - \frac{1}{2m} [i\hbar \nabla_\tau + 2e\mathbf{A}(\mathbf{r})c]^2 \right] G^0(\mathbf{r},\mathbf{r}';i\omega_n) = \delta(\mathbf{r} - \mathbf{r}'). \tag{4}$$

The charge of the bosons is assumed to be $2e$.

As was shown in Ref. 3, a charged Bose gas is a type-II superconductor with a large Ginzburg–Landau coefficient κ. At fields $H \gg H_{c1}$, the nonuniformity of $\mathbf{B}(\mathbf{r}) = \nabla \cdot \mathbf{A}(\mathbf{r})$, which is governed by the penetration depth λ, is small in comparison with the distance between vortices and in comparison with the Larmor radius $l^2 = (1/2eH)$. The field $\mathbf{B}(\mathbf{r})$ can thus be assumed to be constant in Eqs. (2) and (3). All the calculations below are carried out in a system of units with $\hbar = c = k_b = 1$. The Green's functions F and G and also the condensate wave function ξ_0 transform in the following way upon translation by an arbitrary constant vector $\mathbf{a}$

$$G(\mathbf{r},\mathbf{r}';i\omega_n) = \exp[i2e(\mathbf{r} - \mathbf{r}')\mathbf{A}(\mathbf{a})] G(\mathbf{r}+\mathbf{a},\mathbf{r}'+\mathbf{a};i\omega_n), \tag{5}$$

$$F(\mathbf{r},\mathbf{r}';i\omega_n) = \exp[i2e(\mathbf{r} + \mathbf{r}')\mathbf{A}(\mathbf{a})] F(\mathbf{r}+\mathbf{a},\mathbf{r}'+\mathbf{a};i\omega_n), \tag{6}$$

$$\xi_0(\mathbf{r}) = \exp[i2e\mathbf{r}\mathbf{A}(\mathbf{a})] \xi_0(\mathbf{r}+\mathbf{a}). \tag{7}$$

We choose the vector potential $\mathbf{A}(\mathbf{r}) = \mathbf{H} \times \mathbf{r}/2$ with $\mathbf{H} = (0,0,H)$. As in Ref. 4, we use a representation in which the x coordinate of the center of the Larmor orbits is a quantum number:

$$\chi_{LX,p_z}(\mathbf{r}) = \exp\left(\frac{ixy}{2l^2} \right) \frac{\exp[i(-Xy/l^2 + p_z z)]}{\sqrt[4]{\pi l^2} \sqrt{2^L L!}} \exp\left(- \frac{(x-X)^2}{2l^2} \right) H_L\left(\frac{(x-X)}{l} \right), \tag{8}$$

where X is the x coordinate of the center of the orbit, L is the index of the Landau level, and $H_L(x)$ is a Hermite polynomial. The intrinsic energy is

$$\varepsilon_{L,p_z}=(L+1/2)\frac{2eH}{m}+\frac{p_z^2}{2m}.$$

A particle in the periodic potential formed by the vortex lattice of the Bose condensate, (1), can be scattered only into a state which differs from the original state by a reciprocal-lattice vector. Since the x coordinate of the center of the orbit, X, corresponds to the momentum in the y direction, the change in momentum is $\Delta X=2\pi ml^2/a_y$, where m is an integer. In representation (8), the function G thus depends on only the combinations $X_1=X+\pi ml^2/a_y$ and $X_2=X-\pi ml^2/a_y$.

The anomalous Green's function F is zero except in the case in which the centers of the boson orbits satisfy the Aharonov–Bohm interference conditions, $X_1=(ma_x/2M)+X$ and $X_2=(ma_x/2M)-X$. The Green's functions G and F constructed in this manner satisfy periodicity conditions (5) and (6). Using the relation $a_x/M=2\pi l^2/a_y$, we expand G and F in the eigenfunctions in (8):

$$G(\mathbf{r},\mathbf{r}';i\omega_n)=\sum_{L_1,L_2}\sum_m\int\frac{dX}{2\pi l^2}\chi_{L_1,X+ma_x/2M}(\mathbf{r})\chi^*_{L_2,X-ma_x/2M}(\mathbf{r}')G_{L_1,L_2}(X,m;i\omega_n),$$

$$F(\mathbf{r},\mathbf{r}';i\omega_n)=\sum_{L_1,L_2}\sum_m\int\frac{dX}{2\pi l^2}\chi_{L_1,ma_x/2M+X}(\mathbf{r})\chi_{L_2,ma_x/2M-X}(\mathbf{r}')F_{L_1,L_2}(m,X;i\omega_n). \tag{9}$$

To simplify the equations, we have omitted the variable p_z from the expressions for G and F. It follows from the quasiperiodicity conditions, (5) and (6), that we have

$$F_{L_1,L_2}(m+2M,X;i\omega_n)=F_{L_1,L_2}(m,X;i\omega_n),$$
$$G_{L_1,L_2}(X+a_x,m;i\omega_n)=G_{L_1,L_2}(X,m;i\omega_n). \tag{10}$$

In addition, the following relations hold for the Green's functions G and F in a magnetic field:

$$G_{L_1,L_2}(X,m;i\omega_n)=G^*_{L_1,L_2}(X,-m;i\omega_n),$$

$$F_{L_1,L_2}(m,X;i\omega_n)=F_{L_1,L_2}(m,-X;i\omega_n).$$

The quantity M and the ratio a_x/a_y are determined by the geometry of the lattice. For square and triangular lattices we would have $M=1$ and $M=2$respectively.

Let us rewrite Eqs. (2) and (3) in representation (9):

$$G_{L_1,L_2}(X,m;i\omega_n)=G^0_{L_1}(i\omega_n)\delta_{L_1,L_2}\delta_{m,0}+G^0_{L_1}(i\omega_n)\sum_{L'}\sum_{m'}\left\{\Sigma^{11}_{L_1,L'}\left(X-\frac{a_x}{2M}m',m-m';i\omega_n\right)G_{L',L_2}\left[X+\frac{a_x}{2M}(m-m'),m';i\omega_n\right]+\Sigma^{20}_{L_1,L'}\left(m-m',X-\frac{a_x}{2M}m';i\omega_n\right)F_{L',L_2}\left[m',X+\frac{a_x}{2M}(m-m');i\omega_n\right]\right\}, \quad (11)$$

$$F_{L_1,L_2}(m,X;i\omega_n)=G^0_{L_1}(-i\omega_n)\sum_{L'}\sum_{m'}\times\left\{\Sigma^{11}_{L_1,L'}\left(X+\frac{a_x}{2M}m',m-m';i\omega_n\right)F_{L',L_2}\left[m',X-\frac{a_x}{2M}(m-m');i\omega_n\right]+\Sigma^{02}_{L_1,L'}\left(m-m',X+\frac{a_x}{2M}m';i\omega_n\right)G_{L_2,L'}\left[\frac{a_x}{2M}(m-m')-X,m';i\omega_n\right]\right\},$$

where the intrinsic-energy parts Σ^{02} and Σ^{11} are given by

$$\Sigma^{11}_{L_1,L_2}(X,m;i\omega_n)=\int d\mathbf{r}_1 d\mathbf{r}_2 \chi^*_{L_1,X+ma_x/2M}(\mathbf{r}_1)\chi_{L_2,X-ma_x/2M}(\mathbf{r}_2)\Sigma^{11}(\mathbf{r}_1,\mathbf{r}_2;i\omega_n),$$

$$\Sigma^{02}_{L_1,L_2}(m,X;i\omega_n)=\int d\mathbf{r}_1 d\mathbf{r}_2 \chi^*_{L_1,X+ma_x/2M}(\mathbf{r}_1)\chi^*_{L_2,ma_x/2M-X}(\mathbf{r}_2)\Sigma^{02}(\mathbf{r}_1,\mathbf{r}_2;i\omega_n).$$

Using periodicity properties (10) and the parity of the Green's functions G and F, we can diagonalize Eq. (11) with respect to the variable m for a square lattice ($M=1$) by means of the transformation

$$G_{L_1,L_2}(\mathbf{R};i\omega_n)=\sum_m e^{-2\pi imY/a_y}G_{L_1,L_2}\left(X-\frac{a_x}{2M}m,m;i\omega_n\right),$$

$$F_{L_1,L_2}(\mathbf{R};i\omega_n)=e^{-iXY/l^2}\sum_m e^{2\pi imY/a_y}F_{L',L_2}\left(m,\frac{a_x}{2M}m-X;i\omega_n\right). \quad (12)$$

The function $G(\mathbf{R})$ constructed in this manner is periodic, while $F(\mathbf{R})$ is quasiperiodic, in X and Y, with respective periods a_x and a_y. The quantity $\mathbf{R}$ determines the Brillouin zone:

$$G_{L_1,L_2}(\mathbf{R};i\omega_n)=G^0_{L_1}(i\omega_n)\delta_{L_1,L_2}+G^0_{L_1}(i\omega_n)\sum_{L'}[\Sigma^{11}_{L_1,L'}(\mathbf{R};i\omega_n)G_{L',L_2}(\mathbf{R};i\omega_n)+\Sigma^{20}_{L_1,L'}(\mathbf{R};i\omega_n)F_{L',L_2}(\mathbf{R};i\omega_n)],$$

$$F_{L_1,L_2}(\mathbf{R};i\omega_n)=G^0_{L_1}(-i\omega_n)\sum_{L'}[\Sigma^{11}_{L_1,L'}(-\mathbf{R};i\omega_n)F_{L',L_2}(\mathbf{R};i\omega_n)$$

$$+\Sigma^{02}_{L_1,L'}(\mathbf{R};i\omega_n)G_{L',L_2}(\mathbf{R};i\omega_n)]. \tag{13}$$

For fields $H_{c1}\ll H\ll H_{c2}$ we calculate the Green's functions $F(\mathbf{R})=F_{0,0}(\mathbf{R})$ and $G(\mathbf{R})=G_{0,0}(\mathbf{R})$ in the approximation of the zeroth Landau level (the ultraquantum limit). In the calculations we replace the Green's functions G with $L\neq 0$ by expression (4), since the condensate density is low [$(2eH/m\gg 4\pi n_0 a/m$, where n_0 is the density of the condensate, and a is an effective scattering amplitude]:

$$G(\mathbf{R})=\frac{\Sigma^{11}(-\mathbf{R})-G_0^{0-1}(-i\omega_n)}{\mathrm{Det}(\mathbf{R};i\omega_n)},$$

$$F(\mathbf{R})=-\frac{\Sigma^{02}(\mathbf{R})}{\mathrm{Det}(\mathbf{R};i\omega_n)}, \tag{14}$$

where

$$\mathrm{Det}(\mathbf{R};i\omega_n)=\Sigma^{20}(\mathbf{R})\Sigma^{02}(\mathbf{R})-\left[\Sigma^{11}(\mathbf{R})+\frac{p_z^2}{2m}-\mu'-i\omega_n\right]$$

$$\times\left[\Sigma^{11}(-\mathbf{R})+\frac{p_z^2}{2m}-\mu'+i\omega_n\right],$$

$$\Sigma^{20}(\mathbf{R})=\Sigma^{20}_{0,0}(\mathbf{R}),\quad \Sigma^{11}(\mathbf{R})=\Sigma^{11}_{0,0}(\mathbf{R}),\quad \mu'=\mu-\frac{eH}{m}.$$

Here μ is the chemical potential, which is defined in exactly the same way as for an uncharged Bose gas, by the equation

$$\mu'=\Sigma^{11}(0)-\Sigma^{20}(0). \tag{15}$$

Using the replacement $\omega_n\rightarrow -i\omega-\delta$, we find an equation for the spectrum of excitations of the charged Bose gas:

$$\mathrm{Det}(\mathbf{R};\omega)=0. \tag{16}$$

In a real system in which heavy bipolarons may form,[7] there are light electrons, which screen the bosons. Furthermore, in metal oxides the Coulomb part of the interaction between bipolarons is small in comparison with the short-range interaction, because of the large dielectric constant ε_0. As a result, the assumption of a short-range potential between the charged bosons not only simplifies the description but is in fact the most realistic assumption. In the immediate vicinity of the critical point T_{c2}, we can restrict the calculation of the intrinsic-energy parts Σ^{11} and Σ^{20} to the first correction in terms of the condensate density n_0, and we can assume that the total particle density $n=n_0+n'$ is independent of the coordinates. In first-order perturbation theory the intrinsic-energy part Σ^{11} is given by

$$\Sigma^{11}(\mathbf{R})=\frac{2v_0}{\sqrt{2\pi}la_y}\int\frac{d\mathbf{R}'}{2\pi l^2}\exp\left(-\frac{X'^2}{2l^2}\right)\vartheta_3\left(\frac{Y'}{a_y}\middle| i\right)\int\frac{dp_z}{2\pi}n'(\mathbf{R}'-\mathbf{R},p_z)+\Sigma^{11}_{\{n_0\}}(\mathbf{R}), \tag{17}$$

where

$$\Sigma^{11}_{\{n_0\}}(\mathbf{R})=2n_0v_0\exp\left(-\frac{X^2}{2l^2}\right)\vartheta_3\left(i\frac{X}{a_y}\middle| i\right)\vartheta_3\left(-\frac{Y}{a_y}\middle| i\right)$$

for a square lattice [$\vartheta_3(v|\tau)$ are the Jacobi theta functions[8]], and

$$n'(\mathbf{R},p_z)=-T\lim_{\tau\to 0}\sum_{\omega}G(\mathbf{R},p_z;i\omega_{\_n})e^{-i\omega_{\_n}\tau}$$

is the number of particles above the condensate. The integration in (17) over X is between infinite limits, while that over Y is over a period a_y.

Because of the uniformity of the total boson density, the quantity $\Sigma^{11}(\mathbf{R})$ is also independent of the coordinates. To see this, we use the spectrum $\varepsilon(p_z)=J\sqrt{p_z}$ found in Ref. 3 for $\omega_n=0$ at $H=H_{c2}$. Near H_{c2} we consider a region in which the parameter

$$\Lambda=\frac{J(2mJ)^{1/3}}{v_0n_0}$$

is much greater than one. Using the expression found for J in Ref. 3, we then find the following equation for $\sigma(\mathbf{R})=\Sigma^{11}(\mathbf{R})-\Sigma^{11}(\mathbf{0})$:

$$\sigma(\mathbf{R})=\sigma^0(\mathbf{R})+\frac{2l}{\sqrt{2\pi}a_y}\ln\Lambda\int\frac{d\mathbf{R}'}{2\pi l^2}\exp\left(-\frac{X'^2}{2l^2}\right)\vartheta_3\left(\frac{Y'}{a_y}\middle| i\right)\sigma(\mathbf{R}'-\mathbf{R}), \tag{18}$$

where $\sigma^0(\mathbf{R})=\Sigma^{11}_{\{n_0\}}(\mathbf{R})-\Sigma^{11}_{\{n_0\}}(\mathbf{0})$. This equation can be solved analytically, but we will content ourselves with an estimate of σin the limit $\ln\Lambda\gg1$. In this case we have $\sigma\sim(\sigma^0/\ln\Lambda)$, so we can ignore $\Sigma^{11}(\mathbf{R})$ in comparison with the anomalous intrinsic-energy part. The spectrum is then determined by the anomalous intrinsic-energy part $\Sigma^{20}(\mathbf{R})$:

$$\omega^2(\mathbf{R})=\Sigma^{20}(\mathbf{R})\Sigma^{02}(\mathbf{R})-[\Sigma^{20}(0)]^2. \tag{19}$$

For the form of the wave function in (1), we find the following result for a square vortex lattice, using the definition[8] of the Jacobi theta functions:

$$\Sigma^{20}(\mathbf{R})=v_0n_0\exp\left(i\frac{XY}{l^2}-\frac{X^2}{l^2}\right)\vartheta_3^2\left(\frac{Y-iX}{a_y}\middle| i\right), \tag{20}$$

where the lattice constants satisfy $a_x=a_y=l\sqrt{2\pi}$, v_0 is the interaction potential, $n_0=(N_0/V)$ is the average density of the Bose condensate, and we have $\Sigma^{02}(\mathbf{R})=\Sigma^{20*}(-\mathbf{R})$.

Near the bottom of the magnetic Brillouin magnetic zone, the excitation spectrum is linear:

$$\omega(\mathbf{R})=\frac{v_0 n_0}{l}\,\vartheta_3^2(0|i)R, \tag{21}$$

where $R=\sqrt{X^2+Y^2}$.

The anisotropy of spectrum (16) leads to an anisotropy and a nonuniformity of the local density of states of carriers. The density of states in turn determines the tunneling conductivity $\sigma(V,\mathbf{k},\mathbf{r})$, which can be measured with a scanning tunneling microscope.

By analogy with conventional superconductors, we would expect a triangular vortex lattice in a charged Bose gas. However, a study of the type of lattice which is actually realized near H_{c2} goes beyond the scope of this letter.

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