Supersymmetric instanton

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A definition of superinstanton is proposed on the basis of a twistor interpretation of the Yang-Mills supersymmetry equations. A corresponding generalization of the ADHM construction is derived.

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Among the solutions of the Yang-Mills vacuum equations on $\mathbb{R}^4$ (or on $\mathbb{C}^4$, for a holomorphic connectedness) it is natural to single out instantons, i.e., solutions with a self-dual (or an anti-self-dual) curvature form. Here the dimensionality of the $n=4$ phase of the main stratification, in which the connectednesses are specified, plays an important role. The Hodge operator * maps the algebra of external forms onto a base in itself, transforming the p forms into $(n-p)$ forms. Remarkably, with $n=4$ it is possible to write the equation $*F=\pm F$ for the curvature; furthermore, by virtue of the Bianchi identity, any solution of the duality equations also satisfies the Yang-Mills equations. A general solution of the duality equations is given by the ADHM construction.[1,2]

The question of the generalization—in any sense—of the concept of an instanton to the Yang-Mills supersymmetry theory is therefore nontrivial. It would be natural to require that the even-even components of F continued to satisfy the duality condition, but it is not obvious which considerations should guide the choice of conditions on the other components, and it is not obvious whether we should attempt to satisfy the Yang-Mills supersymmetry equations.

The candidate for the role of superinstanton which will be described below was constructed through the use of Witten's treatment[3] of the Yang-Mills supersymmetry equations. Specifically, considering the superspace $(x_{AA'}, \theta_{iA}, \theta^{iA'})$, $i=1,\ldots,N$ and the superconnectedness $(A_{AA'}, \zeta_{Ai}, \zeta_{A'i})$, we define the operators

$$\nabla^{Ai} = \frac{\partial}{\partial\theta_{Ai}} + \theta^{A'i}\partial^A_{A'} + \zeta^{iA}, \tag{1a}$$

$$\nabla_{A'i} = \frac{\partial}{\partial\theta^{A'i}} + \theta_{Ai}\partial^A_{A'} + \zeta_{A'i}, \tag{1b}$$

and we impose on them the conditions

$$\{\nabla_{A'i}, \nabla_{B'j}\} + \{\nabla_{B'i}, \nabla_{A'j}\} = 0, \tag{2a}$$

$$\{\nabla^{Ai}, \nabla^{Bj}\} + \{\nabla^{Bi}, \nabla^{Aj}\} = 0, \tag{2b}$$

$$\{\nabla_{Ai}, \nabla_{A'j}\} = 2\delta_{ij}\nabla_{AA'}, \tag{2c}$$

where $\nabla_{AA'} = \partial_{AA'} + A_{AA'}$. With $N=3$ the superconnectedness satisfies the Yang-Mills supersymmetry equations.[3]

We will assume that the stronger conditions in (2b), (2c), and

$$\{\nabla_{A'i}, \nabla_{B'j}\} = 0 \tag{2d}$$

serve as a definition of the superinstanton.

The reason for this definition is that conditions (2b), (2c), and (2d) lead to the self-duality condition $F_{AB} = 0$, where

$$[\nabla_{AA'}, \nabla_{BB'}] = F_{AA'BB'} = \epsilon_{AB}F_{A'B'} + \epsilon_{A'B'}F_{AB}. \tag{3}$$

To prove this assertion, we note that a consequence of (2b), (2c), and (2d) is

$$\begin{aligned}[\nabla_{A'i}, \nabla_{BB'}] &= (1/2N)\,[\nabla_{A'i}, \{\nabla_{Bj}, \nabla_{B'j}\}] \\ &= (1/2N)([\{\nabla_{A'i}, \nabla_{Bj}\}, \nabla_{B'j}] + [\{\nabla_{A'i}, \nabla_{B'j}\}, \nabla_{Bj}]) \\ &= (-1/N)[\nabla_{B'i}, \nabla_{A'B}] = (1/N^2)[\nabla_{A'i}, \nabla_{BB'}].\end{aligned} \tag{4}$$

With $N > 1$ we thus have

$$[\nabla_{A'i}, \nabla_{BB'}] = 0. \tag{5}$$

Analogously, we find

$$[\nabla_{Ai}, \nabla_{BB'}] + [\nabla_{Bi}, \nabla_{AB'}] = 0. \tag{6}$$

We can now write

$$2F_{AB} = (1/2N)\,\{[\nabla^{B'}_{A}, \nabla_{Bi}], \nabla_{B'i}\}, \tag{7}$$

but by virtue of (6) this expression is antisymmetric with respect to the indices (A, B), while $F_{AB} = F_{BA}$; hence, $F_{AB} = 0$.

The field $A_{AA'}$ may be thought of as a connectedness specified on $\mathbb{R}^4$ (or a holomorphic connectedness on $\mathbb{C}^4$) which has a self-dual curvature form and which depends on certain additional parameters $(\theta_{Ai}, \theta^{A'i})$. In this case, however, $A_{AA'}$ is generated by the

ADHM construction, all of whose ingredients depend on these odd variables. Considering a k instanton in the SU(n) group for definiteness, and adopting the notation of Ref. 2, we thus find

$$A_{AA'}(x,\theta) = v^+(x,\theta)\partial_{AA'}v(x,\theta), \tag{8a}$$

$$v^+(x,\theta)\,\Delta^A(x,\theta) = 0, \quad v^+(x,\theta)v(x,\theta) = 1, \tag{8b}$$

$$\Delta^A(x,\theta) = a^A(\theta) + b_{A'}(\theta)\,x^{A'A}, \tag{8c}$$

$$P = vv^+ = 1 - \Delta^A f \Delta^+_A, \quad \Delta^+_A\,\Delta^B = \delta^B_A f^{-1}. \tag{8d}$$

From considerations of gauge covariance, however, we have

$$\zeta_{Ai} = v^+ D_{Ai} v, \qquad \zeta_{A'i} = v^+ D_{A'i} v, \tag{9}$$

where D_{Ai} and $D_{A'i}$ are found from (1) in the case of a zero connectedness.

To determine the explicit dependence on θ in (8c), we will use (8a) and (9) to rewrite requirements (2b), (2c), and (2d) as follows, respectively:

$$v^+(\{D_{Ai}P, D_{Bj}P\} + \{D_{Bi}P, D_{Aj}P\})v = 0, \tag{10a}$$

$$v^+\{D_{Ai}P, D_{A\,j}P\}\,v = 0, \tag{10b}$$

$$v^+\{D_{A'i}P, D_{B'j}P\}v = 0. \tag{10c}$$

This system of equations is satisfied with

$$\Delta^A(x,\theta) = a^A + b_{A'}(x^{A'A} + \theta^{A'i}\theta^{Ai}) + c_i\theta^{Ai}, \tag{11}$$

where the matrices a^A, $b_{A'}$, and c_i (the latter with odd elements) are independent of θ. In fact, using (8b) and (8d) in calculating v^+dP, we find

$$v^+D_{A'i}P = 0, \tag{12}$$

$$v^+\{D_{Ai}P, D_{Bj}P\}\,v \sim \epsilon_{AB}, \tag{13}$$

which gives us the zero in (10a) upon symmetrization.

The superinstanton defined here thus turns out to be a superinstanton in the sense which follows from the generalization of the ADHM construction. By analogy with the usual case, expression (11) for Δ^A results from an operator which is linear in the supertwistors[4] $(z_A, z^{A'}, \xi^i)$,

$$A(z,\xi) = a^A z_A + b_{A'}z^{A'} + c_i\xi^i, \tag{14}$$

where the supertwistors are connected by the incidentness conditions[4,5]

$$z^{A'} = (x^{A'A} + \theta^{A'i}\theta^{Ai})z_A, \qquad \xi^i = \theta^{Ai}z_A. \tag{15}$$

The similarity of the supersymmetry version of the ADHM construction to the ordinary construction means that we can transfer the entire "instanton technique" developed in a series of studies,[1,2,6] to the supersymmetry version of the construction.

It would also be interesting to attempt to relate the possibility of a supersymmetry generalization of the ADHM construction to the properties of this construction in the even case.

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