Dynamic SO(3,1) symmetry of a Dirac magnetic monopole

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New integrals of motion are derived for a classical charged spin-zero particle in the field of a Dirac magnetic monopole. These integrals of motion generate, along with the angular momentum integrals, the SO(3,1) algebra of Poisson brackets. This symmetry may be related to the structure of the amplitude for the scattering of a quantized particle.

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Over the years since Dirac's paper[1] on the magnetic monopole appeared, this entity has become the subject of a large number of papers (see Refs. 2 and 3 and the bibliography there). We now have a deeper understanding of the topological nature of magnetic charge, of the dynamics of particles in the field of such a charge, and of the structure of the scattering amplitude.[1–3]

One of the interesting questions which has been taken up is the dynamic symmetry of the monopole. In this letter we will derive a new vector integral, **K**, for the motion of a classical nonrelativistic charged particle in the field of a monopole. This integral generates, along with the angular momentum vector, the SO(3,1) algebra of Poisson brackets. This algebra differs from the SO(3)×SO(2,1) algebra found by Jackiw,[8] since the latter contains quantities which do not commute with the Hamiltonian, while the integral **K** does. So far, it has not been possible to construct quantum-mechanical analogs of these integrals. The difficulty is that the complexity of the analytic expressions for the integrals makes the quantum-mechanical dynamic variables noncommuting.

The equations of motion of a classical charged spin-zero particle in the field of a monopole are the Hamiltonian system for the coordinates r_i and the generalized momenta of the particle:

$$P_i = p_i - (eZ/c)A_i\ ,\quad i = 1,2,3$$

where p_i are the momenta, and A_i is the vector potential of the magnetic field of the monopole.[4] Let us examine motion with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\,\frac{p^2}{m}\ .$$

The Poisson brackets for the dynamic variables are

$$[r_i\ , r_j] = 0,\ [r_i\ , P_j] = \delta_{ij},\ [P_i\ , P_j] = \epsilon_{ijk}H_k\ . \qquad (1)$$

Here H_k are the magnetic field components. The angular momentum vector **L** is[4]

$$\mathbf{L} = \mathbf{r}\times\mathbf{P} - q\mathbf{r}/r\ ,$$

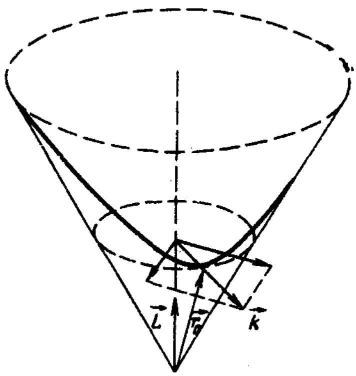

FIG. 1. Trajectory of a classical charged spin-zero particle in the field of a magnetic monopole. The trajectory lies on the cone $\mathbf{L}\mathbf{r} = qr$. Here $\mathbf{r}_0$ is a vector drawn from the vertex of the cone to the nearest point on the trajectory. The vectors $\mathbf{K}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ are orthogonal to the axis of the cone, which is parallel to the angular momentum $\mathbf{L}$. $\mathbf{K}^2 = \mathbf{e}_1^2 = \mathbf{e}_2^2 = L^2 - q^2$.

where $q = eg/c$, and g is the magnetic charge. The following relations hold:

$$[L_i, r_j] = \epsilon_{ijk} r_k, \; [L_i, P_j] = \epsilon_{ijk} P_k, \; [L_i, K_j] = \epsilon_{ijk} L_k . \qquad (2)$$

The integrals of motion are known[4]: the energy of simple P^2, the angular momentum $\mathbf{L}$, and $q = \mathbf{L}\mathbf{n}$, where $\mathbf{n} = \mathbf{r}/r$.

The new integrals which we wish to point out here are constructed in the following way. We introduce a quantity which is an integral of motion (Fig. 1):

$$r_0^2 = P^{-2}(L^2 + q^2) .$$

We then consider the vectors

$$\mathbf{e}_1 = L\,\mathbf{n} - \cos\theta\,\mathbf{L} ,$$

$$\mathbf{e}_2 = \mathbf{L} \times \mathbf{n} ,$$

where $\cos\theta = q/L$, and L is the magnitude of the vector $\mathbf{L}$. The following relations hold:

$$[\mathbf{e}_\mu, P^2] = \epsilon_{\mu\nu} \frac{2L}{r^2} \mathbf{e}_\nu , \; \mu,\nu = 1,2 . \qquad (3)$$

We introduce the vector

$$\mathbf{K} = \cos\tau\,\mathbf{e}_1 + \sin\tau\,\mathbf{e}_2 ,$$

$$\tau = (L/Pr_0)\,\mathrm{arctg}\left(((r/r_0)^2 - 1)^{1/2}\right) .$$

It follows from Eq. (3) that $[\mathbf{K}, P^2] = 0$; i.e., $\mathbf{K}$ is an integral of motion. The following expressions hold:

$$[e^i_\mu, e^j_\mu] = -\epsilon_{ijk} L_k, \quad \mu = 1,2,$$

$$[e^i_1, e^j_2] = L(\delta_{ij} - (\cos\theta / L)^2 L_i L_j), \tag{4}$$

$$[\tau, \mathbf{e}_\mu] = \epsilon_{\mu\nu} \chi \mathbf{e}_\nu, \quad \mu, \nu = 1,2,$$

$$\chi = P^{-2} r_0^{-2} (1 + \cos^2\theta)^{-1} [-\cos^2\theta \operatorname{arctg}((r/r_0)^2 - 1)^{1/2} + (2/r^2)(r^2 - r_0^2)].$$

It follows that

$$[K_i, K_j] = -\epsilon_{ijk} L_k, \quad [L_i, K_j] = \epsilon_{ijm} K_m. \tag{5}$$

It follows from (2) and (5) that the integrals **L** and **K** generate the SO(3,1) Lie algebra of the dynamic symmetry of the monopole.

The procedure outlined in this letter can be extended to the 't Hooft–Polyakov colored monopoles.[5–7]

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[1]P. A. M. Dirac, Proc. R. Soc. **A133**, 60 (1931).
[2]T. T. Wu and C. N. Yang, Nucl. Phys. **B107**, 365 (1976).
[3]J. Schwinger, L. De Raad, K. Milton, D. Clark, and Wu Yang-Tsai, Ann. Phys. (N.Y.) **101**, 451 (1976).
[4]J. Schwinger, Particles, Sources, and Fields, Addison–Wesley, Reading, Mass., 1970 (Russ. transl. Mir, Moscow, 1973).
[5]G. t'Hooft, Nucl. Phys. **B79**, 276 (1974).
[6]A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
[7]E. Corrigan and D. Olive, Nucl. Phys. **B110**, 237 (1976).
[8]R. Jackiw, Ann. Phys. (N.Y.) **129**, 183 (1980).

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