Wu–Yang bundle spaces

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The bundle spaces which arise in the scattering of a charged particle by a Dirac monopole are analyzed. They are shown to be lenslike, and their topology is described.

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Wu and Yang showed[1] that a systematic analysis of the scattering of a charged particle by a monopole requires the techniques of fiber-bundle theory. In the present letter we wish to explain the geometric meaning of the topological number $n = 2e\mu$, which specifies the Wu–Yang bundles. This explanation may prove useful for a qualitative study of the scattering amplitude and in a search for new representations for it. The poor convergence of the expansion of the amplitude in generalized spherical harmonics apparently requires a more thorough use of topological considerations. We will show that the geometry of the problem is different for different values of n. In the case $n = 1$, it is the geometry of a sphere; for $n = 2$ it is that of a projective space; and in general it is that of so-called lenslike spaces.

We recall[1] that the reason for the appearance of the bundles lies in a singularity of the vector potential of the monopole. If this potential is specified by $\mathbf{A}_r = \mathbf{A}_\theta = 0$, $\mathbf{A}_\varphi = (\mu/r)\tan(\theta/2)$, for example then the semiaxis $\theta = \pi$ serves as the fiber of singularities, while if the tangent is replaced by the negative of the cotangent, the same role is played by the semiaxis $\theta = 0$. These two descriptions differ by $2\mu\nabla\varphi(x)$, where φ is the azimuthal angle. The Schrödinger equation for a particle in the field of a monopole also has a singularity if we use a common gauge throughout the space. We may, however, examine a pair of equations, each in a region in which its own vector potential is continuous. We denote these regions by V and V'. In their intersection the corresponding wave functions are related by the gauge transformation

$$\Psi(x) = \Psi'(x)\exp(2ie\mu\varphi(x)), \tag{1}$$

and both are continuous; we then find the Dirac quantization condition for the product of the electric and magnetic charges, $2e\mu = n$ (an integer). Fiber bundles result from the geometric concept of the gluing of the wave functions Ψ and Ψ' into a single continuous surface. Equivalently, we can glue direct products with the gauge group $V\times U(1)$, $V'\times U(1)$, assuming that the pairs (x,g) and (x',g') are equivalent if $x = x'$, $g = g'\exp(in\varphi(x))$. The resulting topological space is called a "principal bundle space." Let us assume that t is a point in it. We set

$$\Phi(t) = \begin{cases} g^{-1}\Psi(x) \\ \\ g'^{-1}\Psi'(x') \end{cases}, \tag{2}$$

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Where both equations in (2) work, they yield identical results. We thus have a unified function, and the previous pair of Schrödinger equations can be replaced by a single equation in the new space.

The gluing function $e^{in\varphi}$ specifies these spaces in an implicit manner. To find an explicit specification we must use a different realization of principal bundles, as bundles into orbits with respect to the free action of the gauge group.[2] The problem is simplified in that the base of Wu–Yang bundles, which is R^3 minus the point occupied by the monopole, is homotopically equivalent to the sphere S^2. It is thus sufficient to examine their compactifications onto a sphere, and they must reduce to a Hopf bundle, which is universal for two-dimensional bases and for the $U(1)$ group. A "Hopf bundle" is a bundle into the orbits of the three-dimensional sphere $S^3 = \{ |z_1|^2 + |z_2|^2 = 1 \}$ in the complex space C^2 with respect to the action of $U(1)$ in accordance with the rule $(z_1,z_2)e^{i\varphi} = (z_1 e^{i\varphi}, z_2 e^{i\varphi})$. Its base is S^2. Specifically, we map S^3 into S^2, associating the number z_1/z_2 with the pair (z_1,z_2), and we then map this number into S^2 through a stereographic projection. By virtue of the continuity, this mapping is continued to $z_2 = 0$. We see that each point in S^2 represents one orbit. It turns out that the Hopf bundle is equivalent ot the Wu–Yang bundle for $n = 1$. To see this, it is sufficient to construct two local cross sections of this bundle, σ and σ', which are coupled by a transformation function that coincides with the gluing function $e^{i\varphi}$. We specify the first cross section by

$$z_1 = \cos\frac{\theta}{2}, \quad z_2 = \sin\frac{\theta}{2} e^{-i\varphi}, \theta \neq \pi . \tag{3}$$

We find the second by setting

$$z_1 = \cos\frac{\theta}{2} e^{i\varphi}, \quad z_2 = \sin\frac{\theta}{2}, \quad \theta \neq 0. \tag{4}$$

Equivalence means, in particular, that the space of Wu–Yang bundles with $n = 1$ is homeomorphic with respect to S^3.

We are left with the problem of constructing spaces for the other topological numbers. This problem can be dealt with by directly applying the universal-bundle construction of Ref. 2, identifying the base S^2 with a complex plane, and performing the inverse mapping of the Hopf bundles with respect to power-law mappings $z \to z^n$. A more convenient representation can be found, however, by taking a different approach. We consider the coupling of the Hopf bundle with the projective space RP^3. This space is found from S^3 by making opposite points identical; i.e., it serves as the factor space S^3/Z_2. We denote its elements by square brackets. To convert RP^3 into the principal bundle, it is sufficient to specify the action of the group $U(1)$ on it. We cannot, however, simply transfer it from S^3, since this is not a free action. Freedom of action means that all the elements of the group except unity must shift each point; only then does it generate a principal bundle. The correct formula is thus

$$[z_1, z_2] e^{i\varphi} = [z_1 e^{i\varphi/2}, z_2 e^{i\varphi/2}]. \tag{5}$$

The cross sections of this bundle are found from the Hopf-bundle cross sections in (3) and (4):

$$[\sigma'] = [\sigma e^{i\varphi}] = [\sigma] e^{2i\varphi} . \tag{6}$$

The transformation function in this case is thus $e^{2i\varphi}$, and RP^3 is homeomorphic with respect to the Wu–Yang bundle space for $n = 2$.

We turn now to the general case. We define the action on S^3 of the cyclic group Z_n by specifying the effect of its generatrix $e^{2\pi i/n}$ by the formula

$$(z_1, z_2) e^{2\pi i/n} = (z_1 e^{2\pi i/n}, z_2 e^{2\pi i/n}). \tag{7}$$

We denote the factor space $S^3 + Z_n$ by L_n^3. Such generalizations of projective space are well known and are referred to as "lenses" or "lenslike spaces." We define the action of the gauge group $U(1)$ on L_n^3 by a formula similar to (5), but with $i\varphi/2$ replaced by $i\varphi/n$. This is a free action, so that L_n^3 transforms into a principal bundle. Its cross sections can also be found from the cross sections of the Hopf bundle, and the transformation function is $e^{in\varphi}$. This result establishes that the lens is homeomorphic with respect to the Wu–Yang bundle space of the same index. We can specify the topological properties of these spaces. The lenses are oriented, and their homotopic groups can be calculated most simply from bundle theory, since the formula $L_n^3 = S^3/Z_n$ also implies a principal bundle, except that in this case a discrete group acts freely, and the orbit consists of n points. Such bundles are referred to as "regular coverings," and it is known[2] that for them the fundamental group of the base coincides with the acting group if the covering space is singly connected. Consequently,

$$\pi_1 (L_n^3) = Z_n . \tag{8}$$

The other homotopic groups are identical for the base and for the covering space,

$$\pi_i (L_n^3) = \pi_i (S^3), \ i \geqslant 2. \tag{9}$$

The nontrivial groups of homologies coincide with homotopic groups, except for the zero-dimensional group, which is equal to Z by virtue of the connectedness of the lens.

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[1]T. T. Wu and C. N. Yang, Phys. Rev. D **12**, 3845 (1975).
[2]B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko. Sovremennaya geometriya (Modern Geometry), Nauka, Moscow, 1979.

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