Efficiency of a gravitational detector with interference of quantum states

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The effect of the initial state and of parametric pumping on the sensitivity of a gravitational detector is discussed in the quantum limit. The possibility of a considerable increase in sensitivity in the parametric resonance regime is demonstrated.

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In recent papers,[1–3] it was pointed out that the energetic response of a quantum oscillator (model of a gravitational detector) to an external force $f(t)$ can be increased by choosing a particular initial state. The arguments supporting the quantum description of a gravitational detector, as is well known,[4,5] are related to the necessity of measuring weak variations in the amplitude of the detector $\Delta x \sim 10^{-18}$–10^{-19} cm, not exceeding the characteristic quantum standard $\Delta x_q \simeq (\hbar/m\omega)^{1/2} \sim 10^{-18}$ cm, with $m \simeq 10^5$ g and $\omega \simeq 10^4$ rad/s.

The purpose of this paper is as follows: 1) to show that such a choice[1,2] does not involve an increase in sensitivity, exceeding previously known estimates; 2) to formulate the search for an optimum initial state of the gravitational detector in general form; and 3) to show that the sensitivity can be increased appreciably by using parametric pumping.

1. The change in the average energy of an oscillator under the action of a resonant force $f(t) = F\sin \omega t$, $\omega t \gtrsim$ is

$$\Delta E(t) = F^2 t^2/8m - Ft\omega < \hat{q}(0) > /2, \tag{1}$$

i.e., for small forces with $\langle \hat{q}(0) \rangle \neq 0$, the contribution of the term linear in the force can be much greater than the quadratic term. For the superposed intitial state $|\psi\rangle = (|n\rangle - |n+1\rangle)/\sqrt{2}$, proposed in Ref. 1 for the purpose of increasing the sensitivity of a gravitational detector, Eq. (1) gives $\Delta E(t) \simeq Ft[\hbar\omega(n+1)/8m]^{1/2}$. For $Ft \ll [m\hbar\omega(n+1)]^{1/2}$, this quantity exceeds the increase in the energy of the oscillator $F^2t^2/8m$, excited from the pure state $|n\rangle$. This difference stems from quantum interference effects. The force $f(t)$ can be recorded when the following inequality is satisfied:

$$|\Delta E| \equiv |E(t) - E(0)| \gtrsim (< \hat{E}^2(0) > - < \hat{E}(0) >^2)^{1/2} \equiv [\sigma_E(0)]^{1/2}. \tag{2}$$

For the superposed state $|\psi\rangle =$, the equality $\sigma_E(0) = (\hbar\omega)^2/4$ is valid; a comparsion of (1) and (2) gives the sensitivity

$$F_{min} \gtrsim 4[m\hbar\omega/(n+1)]^{1/2}/t, \tag{3}$$

which increases indefinitely ($F_{\min}\to 0$) with increasing n. However, the estimate (3) is already contained in Refs. 4 and 5. We shall show that in order to achieve the sensitivity (3), the requirement of a superposed initial state $|\psi\rangle$ is not necessary. For states $|n\rangle$, all centered moments are zero moments, i.e., Eq. (2) is automatically satisfied. The additional requirement, necessary for a single measurement, is that the probability that the state of the detector is conserved is small: $P_{nn}\ll 1$.[6] This condition is satisfied if $g(n+\frac{1}{2})\gtrsim 1$, $g=F^2t^2/8m\hbar\omega$, and it is easy to show that in this case $\mathrm{Sp}[\hat{\rho}_0(\hat{E}(t)-\langle E(t)\rangle)^{2k}]\gtrsim(\hbar\omega)^{2k}$. The opposite assertion, generally speaking, is not necessarily true, but for the variance ($k=1$), it is true, so that if

$$\Delta\sigma_E=\sigma_E(t)=(\hbar\omega/m)(n+1/2)|f_\omega(t)|^2\gtrsim(\hbar\omega)^2, \tag{4}$$

then $P_{nn}\ll 1$ and Eq. (3) is valid. In the above equations, $\hat{\rho}_0$ is the initial density matrix:

$$f_\omega(t)=\int_0^t f(\tau)e^{-i\omega\tau}d\tau.$$

2. In general, the problem of the optimum $\hat{\rho}_0$ (see also Ref. 3) for a given Hermitian observable $\hat{Y}$ reduces to finding states that maximize the functional

$$R\{\hat{\rho},\hat{Y}\}=\{\mathrm{Sp}[\hat{\rho}(\hat{Y}(t,f)-\hat{Y}(t))]\}^2/\mathrm{Sp}\{\hat{\rho}[\hat{Y}(t)-\mathrm{Sp}(\hat{\rho}\hat{Y}(t))]^2\}; \tag{5}$$

$\hat{Y}(t,f)$ and $\hat{Y}(t)$ are Heisenberg operators in the presence of the force and without it. For pure states of the type $|\phi\rangle=\alpha|n\rangle+\beta|n+1\rangle$, $|\alpha|^2+|\beta|^2=1$, using (2), the maximum $|\Delta E|/\sqrt{\sigma_E(0)}$ is attained precisely for the choice $\alpha=-\beta=1/\sqrt{2}$. In the case of correlated states with the Wigner function

$$W(q,p)=\Delta^{-1/2}\exp\{-[\sigma_p\tilde{q}^2+\sigma_q\tilde{p}^2-2\sigma_{pq}\tilde{p}\tilde{q}]/2\Delta\}, \tag{6}$$

$$\tilde{q}=q-\langle\hat{q}\rangle,\quad \tilde{p}=p-\langle\hat{p}\rangle,\quad \Delta=\sigma_p\sigma_q-\sigma_{pq}^2\geqslant\hbar^2/4.$$

the minimum in the variance of the oscillator energy corresponds to $\sigma_{pq}=\langle\hat{p}\rangle=0$ and $\sigma_p\sigma_q=\hbar^2/4$. The observed force in this case is equal to $F_{\min}\gtrsim t^{-1}2m\omega\sqrt{\sigma_q}$ ($[\langle\hat{q}\rangle\lesssim q_*=(\hbar/2m\omega)^2\sigma^{-3/2}]$). Introducing the effective number of quanta $n_*=\langle E\rangle/\hbar\omega\simeq m\omega q_*^2/2\hbar(\sigma_q\ll\hbar/2m\omega,\langle\hat{q}\rangle\simeq q_*)$, we obtain

$$F^*_{\min}\gtrsim t^{-1}(m\hbar\omega)^{1/2}n_*^{-1/6}, \tag{7}$$

i.e., the states (6) lose their advantage to the states $|n\rangle$ or $|\psi\rangle$ when the energy is the observable. However, when the integral of motion $\hat{I}=\hat{q}\cos\omega t-(m\omega)^{-1}\hat{p}\sin\omega t$ is measured (operators of the initial coordinates[5,7]), Gaussian states likewise give sensitivity (3) with their own equivalent value $n_*=\langle\hat{E}\rangle/\hbar\omega$.

3. In order to regenerate the detector parametrically with variable frequency $\omega(t)$,[8] the sensitivity can be increased compared to (3), if the integral of motion $\hat{K}(t)$, which is a generalization of the operator of the quanta, is measured[9]:

$$\hat{K}(t) = (1/2)\{|\epsilon(t)|^2\hat{p}^2 + |\dot{\epsilon}(t)|^2\hat{q}^2 - \mathrm{Re}(\dot{\epsilon}\epsilon^*)(\hat{q}\hat{p}+\hat{p}\hat{q})\}\ ; \tag{8}$$

here $\epsilon(t)$ is the complex solution of the equation $\ddot{\epsilon} + \omega^2(t)\epsilon = 0$, satisfying the condition $\mathrm{Im}(\dot{\epsilon}\epsilon^*) = 1$; the spectrum $\hat{K}(t)$ is an equidistant spectrum: $K_n = n + 1/2$, $n = 0,1,2,...$; we assume that the initial state is a characteristic state for $\hat{K}(t)$. [The difficulties in realizing measurements of $\hat{K}(t)$ are the same as those for the integral of motion $\hat{I}(t)^5$.] Under the action of a force, the operator (8) is no longer an integral of the motion and its variance depends on time:

$$\sigma_K(t) = (n + 1/2)|\delta(t)|^2; \qquad \delta(t) = \int_0^t \epsilon(\tau)f(\tau)d\tau\,. \tag{9}$$

The condition for detection, in analogy with (4), is $\sigma_K \gtrsim 1$. In the case of a resonance in the first Mathieu zone ($[\omega^2(t) = \omega_0^2[1 + 4\mu\cos 2\omega_0 t]; |\mu| \ll 1]$), an approximate solution is valid for $\epsilon(t)$: $= \omega_0^{-1/2}[\mathrm{ch}(\omega_0\mu t)\exp(i\omega_0 t) - i\mathrm{sh}(\omega_0\mu t)\exp(-i\omega_0 t)]$, which for $\omega_0\mu t \gg 1$ leads to the following estimate of the observed force:

$$F_{\min} \gtrsim 2(\omega_0\mu t)\exp(-\omega_0\mu t)(m\hbar\omega_0)^{1/2}/t(n + 1/2)^{1/2}. \tag{10}$$

The gain with $\omega_0\mu t\exp(-\omega_0\mu t) \ll 1$ is due to the increase in the equivalent level n_* as a result of regeneration. The sensitivity (10) is also attained for initial states of the type $|\psi\rangle$ and for Gaussian states, if the integral of motion $\hat{A}(t) = \epsilon(t)\hat{p} - m\dot{\epsilon}(t)\hat{q}$, which is a generalization of the operator $\hat{I}$ to the case of variable frequency $\omega(t)$, is measured. The gain, predicted by (10), occurs only for sufficiently long actions on the gravitational detector. For this reason, we shall estimate the outlook for using a parametric regenerative detector for receiving radiation from pulsars.

For the pulsar in the Crab Nebula, the limits for the predicted energy flux density form the interval $I_0 = 10^{-8}$–10^{-22} erg/cm^2 s,[10] which corresponds to a force $F_{\mathrm{gr}} = (16\pi G c^{-3} I_0)^{1/2} m\omega l$ perturbing the detector; for $m \simeq 10^5$, it corresponds to the length $l = 10^2$ cm, $\omega = 4\times10^2$ rad/s; $F_{\mathrm{gr}} \sim (10^{-13} - 10^{-20})$ dynes. At the same time, the quantum limit to the sensitivity of a Weber detector when measuring its energy in a state with small $n \sim 1$ is of the order of $F_{\mathrm{gr}} \simeq t^{-1}(m\hbar\omega)^{1/2} \sim 10^{-13}$ dynes over times $t \simeq 10^4$ s, less than the relaxation times $\tau^* \sim Q\omega^{-1} \sim 10^6$ s.

A detector of the type in Ref. 11 with a piezo- or magneto- strictive elastic element, pressed between two specimen masses, can be regenerated due to the weak variation in the elastic constants $\mu = \Delta\epsilon/\epsilon \sim 3\times10^{-6}$, controlled by an external field. This gives the factor $\mu\omega t \simeq 12$, and the corresponding detected force F_g (10) decreases by a factor of 10^5 (the unattainable two orders of magnitude are compensated for by the initial excitation with $n \sim 10^5$–10^6). For a shaped detector with a capacitive parametric transducer[12] it is convenient to vary the frequency of the detector through variations of the electromagnetic rigidity, introduced by the sensor. The latter can be comparable to $m\omega_0^2 \sim 10^{10}$ dynes (see details in Ref. 13). As a result, $\mu \sim 10^{-2}$–10^{-4} are easily attainable. The limitations here are related only to the requirement that the contributed damping is small. The difficulties in creating a state of the antenna that is characteristic of the quadratic integral of motion (8) are the same as those for creating a state with fixed energy, since the physical meaning of this quantity is that it is equal to the initial energy of the oscillator. There are no fundamental restrictions on measuring this quantity: This is a linear combination of kinetic and potential energies of the oscillator with time-dependent coefficients. Similarly, the real and imaginary parts of the linear integral of motion $\hat{A} = i2^{-1/2}(\epsilon\hat{p} - \dot{\epsilon}\hat{x})$, found in Ref. 14, which is a generalization of the complex amplitude (see Ref. 5) to the case of a parametric oscillator, can be measured in the same way as the complex amplitude is measured, as discussed in the literature.

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