Axially symmetrical solutions of the two-dimensional Heisenberg model

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It is shown that the axially symmetrical solutions of the Heisenberg model in two-dimensional space can be described by the method of the inverse problem. Infinite series are found for the nonlocal conservation laws and exact solutions. The absence of spatially localized solutions is proved.

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The equation of the Heisenberg model that describes the motion of the magnetization vector **S** in a magnetic field **H** has the form

$$\mathbf{S}_t = [\mathbf{S}, \Delta \mathbf{S}] + [\mathbf{H}, \mathbf{S}]. \tag{1}$$

This equation was investigated in detail for the one-dimensional case in Refs. 1 and 2 by the method of the inverse scattering problem. For the case with a larger number of spatial variables, this equation apparently is not integrable. We shall show that in the two-dimensional case, assuming that the solution is axially symmetrical, the method of the inverse problem is applicable to this model, which permits describing its solution with the same completeness as in the one-dimensional case.

1. Let x_1 and x_2 be Cartesian coordinates in the plane. We shall assume that **S** depends only on t and $x = (x_1^2 + x_2^2)/4$. In this case Eq. (1) can be written in the form

$$\begin{aligned} &A_t - B_x + [A, B] = 0, \\ &[\sigma_3, B] + 2i\,(xA)_x = 0, \end{aligned} \tag{2}$$

where

$$A = g_x g^{-1},\ B = g_t g^{-1},\ 2iS = \vec{\sigma}\mathbf{S} = e^{-(i/2)tH\sigma_3} g^{-1}\sigma_3 g e^{(i/2)tH\sigma_3},$$

and g is chosen so that $g^+g = I$ and diag $A = 0$.

The system of equations (2) comprises the consistency conditions for two linear equations for the matrix ψ:

$$\frac{\partial}{\partial x}\psi = (i\lambda\sigma_3 + A)\psi, \tag{3}$$

$$\frac{\partial}{\partial t}\psi = (2i\lambda^2 x\sigma_3 + 2\lambda x A + B)\psi, \tag{4}$$

where $\lambda = -[2(t+u)]^{-1}$ and u is a spectral parameter in the plane $\mathbb{C}$.

2. We shall first calculate the conservation laws for Eq. (2). We shall assume that $S(x, t_0)$ differs considerably from $\sigma_3/2i$ only in a bounded region with characteristic size R_0. The fundamental solution of the system (3) and (4) can be represented in the form

$$\psi = g(x,t)\chi(x,t,u)\exp(i\lambda\sigma_3 x),$$

where

$$\chi(\infty, t, u) = I.$$

From (3) and (4) it follows that χ satisfies the equations

$$\frac{\partial}{\partial x}\chi = -2\lambda S\chi - i\lambda\chi\sigma_3,\quad \frac{\partial}{\partial t}\chi(0,t,u) = 0 \tag{5}$$

and can be expanded in the series

$$\chi(x,t,u) = I + \sum_{n\geqslant 1}\chi_n(x,t)u^{-n}.$$

The coefficients $\chi_n(x,t)$ are easily calculated and at $x = 0$ do not depend on time. The first two integrals $I_n = \chi_n(0,t)$ have the form

$$I_1 = \int_0^\infty (S(x,t) - \sigma_3/2i)\,dx$$

$$I_2 = tI_1 + \int_0^\infty\int_0^\infty dx_1 dx_2\,\theta(x_2 - x_1)\,[S(x_1,t)S(x_2,t) - (S(x_1,t) + S(x_2,t))\sigma_3/2i - 1/4]. \tag{6}$$

The integral I_1 has the meaning of a magnetic moment. We shall examine the integral

$$J = \mathrm{tr}(I_2 \sigma_3/2i) = (t/2)\int_0^\infty (1 - S_3)\, dx + \int_0^\infty \int_0^\infty dx_1 dx_2\, \theta(x_2 - x_1)\, [\mathbf{S}(x_1, t), \mathbf{S}(x_2, t)]_3 .$$

The first term is positive definite and increases linearly with time, so that the localization radius of the solution must increase with time, since otherwise the second term would remain bounded. In other words, "magnon-drop"-type solutions do not exist. Any fluctuation in the initially localized magnetization, which depends only on the distance to the origin of coordinates, spreads out. Of course, we assume that the integrals I_1 and I_2 converge.

3. The solution of the Cauchy problem reduces to the direct and inverse scattering problems for the operator (3) on the semiaxis $x > 0$. We shall assume that the initial data $A(x, 0)$ are bounded, together with the derivative, and are integrable $(\int_0^\infty |A|\, dx < \infty)$.

We shall examine two sets of fundamental solutions Φ_0 and Φ_+ of Eq. (3), determined by the following conditions:

$$\Phi_0(x, t, \lambda)|_{x=0} = I , \quad \Phi_+(x, t, \lambda) \to \exp(i\lambda\sigma_3 x) \text{ for } x \to +\infty . \tag{7}$$

These solutions are linearly dependent

$$\Phi_0(x, t, \lambda) = \Phi_+(x, t, \lambda)\, T(t, \lambda) \tag{8}$$

and the matrix $T(t, \lambda)$ is called the scattering matrix. Using Eq. (4), we can show that

$$T(t, \lambda) = T\left(0, \frac{\lambda}{1 + 2t\lambda}\right) T^{-1}\left(0, \frac{1}{2t}\right). \tag{9}$$

In other words, $T(t, \lambda)$ is completely determined by its initial value. The matrix $T(t, \lambda)$ carries all information concerning the potential A. Reconstruction of the potential $A(x, t)$ can be reduced to a Weiner–Hopf equation. Of course, the matrix T and the functions Φ_0 and Φ_+ have certain analytical properties and special asymptotic behavior, which we shall use in deriving this equation, but cannot present here. Let $\hat{F}(t, \lambda) = T - I$ and $\hat{f} = \Phi_+ \exp(-i\lambda\sigma_3 x) - I$. Then, the Fourier transformation

$$\begin{aligned} F(t, \xi) &= \int_{-\infty}^{\infty} e^{i\lambda\sigma_3\xi}\, \hat{F}(t, \lambda)\, \frac{d\lambda}{2\pi} , \\ f(x, t, \xi) &= \int_{-\infty}^{\infty} \hat{f}(x, t, \lambda)\, e^{-i\lambda\sigma_3\xi}\, \frac{d\lambda}{2\pi} \end{aligned} \tag{10}$$

is cut-off for $\xi < 0$, while for $\xi > 0$ it satisfies the equation

$$\begin{aligned} &f(x, t, \xi) + \int_\xi^\infty f(x, t, \xi')\, F_1(t, \xi' - \xi)\, d\xi' \\ &+ \int_0^\infty f(x, t, \xi') F_2(t, 2x + \xi + \xi')\, d\xi' + F_2(t, 2x + \xi) = 0 , \end{aligned} \tag{11}$$

where $F_1(t, \xi)$ and $F_2(t, \xi)$ represent the diagonal and antidiagonal parts of the matrix F. The function $g(x, t)$ is reconstructed from the solution of this equation

$$g(x, t) = \Phi_+(x, t, \lambda)|_{\lambda=0} = I + \int_0^\infty f(x, t, \xi)\, d\xi. \tag{12}$$

We note that at the point $x = 0$, the function $g(0, t)$ is determined explicitly from $T(0, \lambda)$ without solving Eq. (11):

$$g(0, t) = T(0, {}^1/_{2t})\, g(0, 0). \tag{13}$$

4. Equations (9) and (11) make it possible to examine the asymptotic behavior of the solution of the Cauchy problem for $|t| \to \infty$ and fixed x. It follows from (9) that $\widehat{F}(t, \lambda) \to 0$ for $t \to \infty$ and

$$F(t, \xi) = t^{-2} \exp(-i\sigma_3 \xi / 2t) F_0 + O(t^{-3}),\ F_0 = \text{const}. \tag{14}$$

Solving (11) in the zeroth-order approximation and substituting it into (12), we obtain

$$2iS = \sigma_3 + t^{-1} e^{-i\sigma_3 \frac{x}{t} - \frac{i}{2} tH\sigma_3} [\sigma_3, g_0] e^{\frac{i}{2} tH\sigma_3} + O(t^{-2}),\ g_0 = \text{const}. \tag{15}$$

Thus, for finite x and $|t| \to \infty$, the solution approaches the unperturbed solution and at the same time the rate and amplitude of precession decrease, consistent with the conclusion arrived at in Sec. 2.

5. The method of the inverse problem allows constructing a series of exact solutions of Eq. (2) using a standard method.[3,4] The wave function $\psi(x, t, \lambda)$ for the N soliton solution is represented in the form

$$\psi_N = \prod_{i \leqslant N} \left(I + \frac{\lambda_i - \bar{\lambda}_i}{\lambda - \lambda_i} P_i(x, t) \right) \exp(i\lambda\sigma_3 x), \tag{16}$$

where

$$P_i^{\alpha\beta}(x, t) = \frac{n_i^\alpha(x, t)\, \bar{n}_i^\beta(x, t)}{\sum_\gamma |n_i^\gamma(x, t)|^2}, \tag{17}$$

$$n_i(x, t) = \psi_{i-1}(x, t, \bar{\lambda}_i) n_{i0},\ \lambda_i = -\frac{1}{2(t + u_i)}.$$

The complex numbers u_i and n_{i0}^1 / n_{i0}^2 are arbitrary parameters of the solution. From the value of $\psi(x, t, \lambda)$ at $\lambda = 0$, it is easy to find $S(x, t)$ in the simplest case of a single-soliton solution

$$S_1 + iS_2 = \frac{2a}{t^2 + a^2} (-it + a \operatorname{th} \varphi) \, e^{iX} / \operatorname{ch} \varphi ,$$

$$S_3 = 1 - \frac{2a^2}{t^2 + a^2} \frac{1}{\operatorname{ch} \varphi} , \tag{18}$$

$$X = \frac{tx}{t^2 + a^2} + p, \quad \varphi = \frac{ax}{t^2 + a^2} + q ,$$

where a, p, and q are arbitrary, real parameters. This solution illustrates the conclusions arrived at in Secs. 2 and 4 concerning the spreading of localized perturbations. Solitons represent a ring wave, whose radius and thickness increase linearly with time (for large t), while the amplitude decreases like t^{-1} for fixed x.

6. In a number of papers (see, for example, Ref. 5), a term describing the anisotropy of the interaction is added to Eq. (1), and in addition to the axial symmetry, it is assumed that the solution has a simple time dependence (uniform precession)

$$g = \exp(-i\sigma_2 \theta(r)) \exp(\omega t \, \sigma_3 / 2i) . \tag{19}$$

In this case, "magnon-drop"-type solutions which decrease rapidly with r for drop sizes greater than some characteristic R_0, were found. In the isotropic case, $R_0 = \infty$, while $\theta(r)$ satisfies the equation

$$\frac{\partial^2}{\partial r^2} \theta + \frac{1}{r} \cdot \frac{\partial}{\partial r} \theta + \omega \sin \theta = 0 , \tag{20}$$

which has nonsingular solutions that decrease in space. For large r, these solutions have the asymptotic form

$$\theta = \frac{\theta_0}{\sqrt{r}} \cos \left(\frac{r}{\sqrt{\omega}} - \theta_0^2 \ln r + c_1 \right) + O(r^{-3/2}) . \tag{21}$$

Because of the slow decrease of the solution, the integrals of the energy and of the magnetic moment diverge and, in addition, it is not integrable in the sense of the norm introduced in Sec. 3; for this reason, it does not fall into the class being studied. The scattering matrix $T(0, \lambda)$ in this case has an essential singularity

$$T(0, \lambda) = g_0 \exp(i\sigma_3 \omega / 4\lambda) g_0^{-1},$$

which must be taken into account in deriving the equations of the inverse problem. In conclusion, we thank É. I. Rashba for a discussion of the results.

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