Complete integrability of the quasi-one-dimensional quantum model of Dicke superradiance

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Using the method of the quantum inverse scattering problem, it is shown that the quasi-one-dimensional quantum model of superradiance is exactly integrable. The commutation relations are obtained for the transition matrix elements and the eigenfunctions and eigenvalues of the integrals of motion of the model are found.

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1. In 1954, Dicke[1] predicted the effect of collective spontaneous emission by a system of two-level atoms, interacting through the transverse electromagnetic field [Dicke superradiance (SR)].

At present, the system of Bloch-Maxwell equations for the slow envelopes of the electric-field intensity, $\epsilon(x, t)$, polarization $p(x, t)$, and population $n(x, t)$ is widely used to describe the SR of extended systems with large Fresnel numbers $F = S/l\lambda_0 \gg 1$, where S is the area of the transverse cross section of the specimen, l is its length, and λ_0 is the characteristic wavelength of the radiation (Ref. 2; see also the review in Ref. 3 and the references cited therein). In dimensionless variables ($\hbar = c = l/N_0 = 1$, where N_0 is the total number of two-level atoms), this system of equations has the form

$$i(\epsilon_t + \epsilon_x) = -\sqrt{\kappa}\, p, \qquad ip_t = \sqrt{\kappa}\, \epsilon\, n$$
$$in_t = 2\sqrt{\kappa}\,(\epsilon^+ p - p^+ \epsilon), \qquad \kappa = 2\pi\omega_{12} d^2 / S, \tag{1}$$

where ω_{12} is the frequency, while d is the transition dipole moment of the atom. At the same time, it is assumed that the variables ϵ, p, and n are classical functions, while quantum effects are included, which is fundamentally necessary in order to describe the spontaneous-emission processes, by introducing into the equation for the function $p(x, t)$ a source of fluctuations ξ_p. It is also well known[4] that the classical equations (1) (without the source of fluctuations ξ_p) admit an exact solution of the Cauchy problem by the inverse scattering problem method.[5]

In this paper, we shall view the quantities ϵ, p, and n as operators with the following commutation relations at coinciding times:

$$[\epsilon(x), \epsilon^+(y)] = \delta(x-y), \qquad [p(x), n(y)] = 2p(x)\delta(x-y)$$
$$[p^+(x), p(y)] = n(x)\,\delta(x-y), \qquad [n(x), p^+(y)] = 2p^+(x)\,\delta(x-y) \tag{2}$$

and we shall show here that the Bloch-Maxwell operator evolution equations (1) and the commutation relations (2) define a Hamiltonian system, which is exactly integrable in the quantum case as well.

2. We shall apply the inverse scattering problem method, developed in Refs. 6–10 for a number of exactly soluble models in quantum field theory, to the system (1) and (2). We shall introduce into the analysis of a two-component fermion field $\psi_\nu(x)$, $\nu = 1.2$, which is related to the operators $p(x)$, $n(x)$ by the relations

$$p(x) = \psi_1^+(x)\,\psi_2(x), \qquad n(x) = \psi_2^+(x)\,\psi_2(x) - \psi_1^+(x)\,\psi_1(x)\,. \tag{3}$$

The operators $\psi_\nu^+(\psi_\nu)$ are essentially field operators for the creation (annihilation) of an electron in the lower ($\nu = 1$) and upper ($\nu = 2$) states of a two-level atom and are related to each other by the completeness conditions: $\psi_2^+(x)\psi_2(x) + \psi_1^+(x)\psi_1(x) = 1$.

We shall examine the auxiliary quantum spectral problem on the finite interval $-L \leqslant x \leqslant L$:

$$\frac{d}{dx}\Phi(x, \lambda) = :Q(x, \lambda)\,\Phi(x, \lambda): \tag{4}$$

where the matrix $Q(x, \lambda)$ has the form

$$Q = i\begin{pmatrix} \frac{1}{2}\left(\lambda - \frac{\kappa}{\lambda}\right) & \sqrt{\kappa}\,\epsilon^+ \\ \sqrt{\kappa}\,\epsilon & -\frac{1}{2}\left(\lambda - \frac{\kappa}{\lambda}\right) \end{pmatrix} + \frac{i\kappa}{\lambda}\begin{pmatrix} \psi_2^+\psi_2 & \psi_2^+\psi_1 \\ \psi_1^+\psi_2 & -\psi_2^+\psi_2 \end{pmatrix}, \tag{5}$$

while the symbol :: indicates normal ordering of the operators in (4). We shall define the matrix $G(x, \lambda)$ as the solution of (4) satisfying the boundary condition $G(x = -L, \lambda) = I$, where I is a unit matrix. Thus the transition matrix for a finite interval is by definition $T_L(\lambda) = G(x = L, \lambda)$. The quantum inverse problem method allows establishing the commutation relations for the matrix elements $T_L(\lambda)$:

$$R_L(\lambda,\mu)T_L(\lambda)\otimes T_L(\mu)=T_L(\mu)\otimes T_L(\lambda)R_L(\lambda,\mu), \tag{6}$$

where, as the corresponding calculations show, the matrix $R_L(\lambda,\mu)$ in our model completely coincides with the R_L matrix in the nonlinear Schrödinger equation model (*N.S.* model).[6–10] This coincidence remains also with the transition to the problem on an infinite interval $-\infty<x<\infty$, where the commutation relations, as before, have the form (6), while the matrices $T(\lambda)$ and $R(\lambda,\mu)$ can be written in the form

$$T(\lambda)=\begin{pmatrix} A^+(\lambda) & B(\lambda) \\ -B^+(\lambda) & A(\lambda)\end{pmatrix} \qquad R(\lambda,\mu)=\begin{pmatrix} 1&0&0&0\\ 0&0&a&0\\ 0&\gamma&0&0\\ 0&0&0&1\end{pmatrix} \tag{7}$$

$$a=\frac{\lambda-\mu}{\lambda-\mu-i\kappa},\qquad \gamma=\frac{\lambda-\mu+i\kappa}{\lambda-\mu+i\delta},\quad \delta\to+0.$$

The existence of the R matrix and the commutation relations obtained with its help for the transition-matrix elements prove that the quantum model of the SR theory (1) and (2) admits a transformation from local fields $\epsilon(x)$, $\psi_\nu(x)$ to action-angle-type variables $A(\lambda)$, $B(\lambda)$ and, therefore, is exactly integrable; in addition, the commutation relations for the elements of the T matrix in our model and in the *N.S.* model coincide completely.

3. Let us now go on to examine the eigenfunctions and eigenvalues of the commutating integrals of motion of the model. As usual for the inverse problem method,[6–10] the operator $A(\lambda)$ does not depend on time, while the operator $\ln A(\lambda)$, which is the generation function of integrals of motion of the system, can be represented as a series in inverse powers of the spectral parameter $(i\lambda)$

$$\ln A(\lambda)=\sum_{n=1}^{\infty}a_n(i\lambda)^{-n}, \tag{8}$$

where the coefficients a_n are commutating integrals of motion of the model and, in particular,

$$a_1=\kappa\int_{-\infty}^{\infty}dx\,[e^+(x)\,\epsilon(x)+\psi_2^+(x)\,\psi_2(x)]\equiv\kappa N, \tag{9}$$

$$a_2=i\kappa\ \{i\int_{-\infty}^{\infty}dx\,\epsilon^+(x)\,\partial_x\,\epsilon(x)+\kappa\int_{-\infty}^{\infty}dx\,[\psi_1^+(x)\,\epsilon^+(x)\,\psi_2(x)$$

$$+\psi_2^+(x)\,\epsilon(x)\,\psi_1(x)]\}\equiv-i\kappa H. \tag{10}$$

The operator N is the number-of-quasiparticles operator, while H is the Hamiltonian of the model. The single-particle state $|\Psi(\mu)\rangle$ is generated by the action of the operator $B(\mu)$ on the vacuum $|0\rangle$, which we take to mean in this case the absence of photons and location of all two-level atoms in the ground state. From the commutation relations (6) and (7), and Eqs. (8)–(10), we find

$$N|\Psi(\mu)\rangle = |\Psi(\mu)\rangle, \quad H|\Psi(\mu)\rangle = -\mu|\Psi(\mu)\rangle. \tag{11}$$

We see from these expressions that the parameter $\omega = -\mu$ is the energy of the single-particle state. Retaining in the operator $B(\mu)$ only the terms giving a nonzero value when operating on the vacuum, we have

$$|\Psi(\mu)\rangle = B(\mu)|0\rangle = i\sqrt{\kappa}\int_{-\infty}^{\infty} dx\, e^{ikx}\left[\epsilon^{+}(x) + \frac{\sqrt{\kappa}}{\mu}\psi_2^{+}(x)\,\psi_1(x)\right]|0\rangle, \tag{12}$$

where $k_1 = \kappa/\mu - \mu$ plays the role of the quasiparticle momentum. The spectrum of single-particle states consists, as expected, of two polariton branches:

$$\omega_1(k) = \frac{k}{2} - \sqrt{\left(\frac{k}{2}\right)^2 + \kappa}, \qquad \omega_2(k) = \frac{k}{2} + \sqrt{\left(\frac{k}{2}\right)^2 + \kappa}. \tag{13}$$

Just as in the *N.S.* model with attraction, in our model, there is a bound state of m particles

$$|\Psi_m(\mu)\rangle = B(\mu_1)\,B(\mu_2)\,...B(\mu_m)|0\rangle, \tag{14}$$

where

$$\mu_l = \frac{\mu}{m} + i\kappa\left(\frac{m+1}{2} - l\right), \quad \mu = \sum_{l=1}^{m}\mu_l, \quad l = 1, 2, ..., m. \tag{15}$$

The energy of the bound m-particle complex per particle $\omega = -\mu/m$, is related to the momentum of the complex as a whole per particle

$$k = \frac{1}{m}\sum_{l=1}^{m}\left(\frac{\kappa}{\mu_l} - \mu_l\right)$$

by the dispersion relation

$$k = \omega\left[1 - \frac{\kappa}{m}\sum_{l=1}^{m}\frac{1}{\omega^2 + \kappa^2\left(\frac{m+1}{2} - l\right)^2}\right]. \tag{16}$$

For sufficiently large $m \gg 1$, the spectrum of the bound state (quantum soliton) is linear $\omega = k$ with a small correction proportional to m^{-1}.

This state, which has the form of a pulse with spatial extent $r_0 \sim (\kappa m)^{-1}$, should be identified as the Dicke superradiance pulse.

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[1]R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[2]J. C. MacGillivray and M. S. Feld, Phys. Rev. A **14**, 1169 (1976).

[3]A. V. Andreev, V. I. Emel'yanov, and Yu. A. Il'inskiĭ, Usp. Fiz. Nauk **131**, 653 (1980) [Sov. Phys. Usp. **23**, 493 (1980)].
[4]G. L. Lamb, Jr., Phys. Rev. Lett. **31**, 196 (1973).
[5]V. E. Zakharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevskiĭ, Teoriya solitonov: metod obratnoĭ zadachi [Theory of Solitons: Method of the Inverse Problem], Nauka, Moscow, 1980.
[6]E. K. Sklyanin and L. D. Faddeev, Dokl. Akad. Nauk SSSR **243**, 1430 (1978).
[7]E. K. Sklyanin, Dokl. Akad. Nauk SSSR **244**, 1337 (1979) [Sov. Phys. Dokl. **24**, 107 (1979)].
[8]H. B. Thacker and D. Wilkinson, Phys. Rev. D **19**, 3660 (1979).
[9]L. D. Faddeev, Preprint LOMI R-2-79, Leningrad, 1979.
[10]H. B. Thacker, Rev. Mod. Phys. **53**, 253 1981.

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