Nonintegrability of the classical Yang-Mills fields

E. S. Nikolaevskiĭ and L. N. Shur
Institute of Theoretical and Experimental Physics; L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 5 August 1982)
Pis'ma Zh. Eksp. Teor. Fiz. **36**, No. 5, 176–179 (5 September 1982)

It is shown for the particular case of the classical Yang-Mills equations that there is no additional first integral. It is concluded that the original system of equations does not have a complete set of integrals.

PACS numbers: 11.15.Kc

1. The question of the integrability of the classical Yang-Mills equations is exceedingly important for both classical and quantum field theory. Interest was attracted to this question by Refs. 1 and 2. Belavin and Zakharov[1] showed that the self-duality equations are the condition for the integrability of the classical Yang-Mills fields; this result was used in Ref. 3 to calculate multi-instanton solutions. Atiyah and Ward and Zakharov *et al.*[2] found that there is no randomization of the initial condition of the Klein-Gordon equation with a cubic nonlinearity—a special case of a Yang-Mills field. Another interesting fact is the nontrivial analogy between the Yang-Mills field and the n field.[4] Both theories are renormalizable, have asymptotic freedom, and have instanton solutions. The equations of the n field, on the other hand, are in integrable system. Under the circumstances it was hoped that the Yang-Mills equations would also be integrable. In this letter we show that, on the contrary, the system of equations describing the classical Yang-Mills fields is nonintegrable.

In the classical case, integrability means that there are integrals of motion, whose Poisson brackets with the energy integral vanish, so that there is the possibility in principle of deriving solutions. From the integrability of the classical problem follows the integrability of the quantum problem[5] in the sense that complete set of commuting operators exists.

2. The Yang-Mills equations are

$$\partial_\mu F^a_{\mu\nu} + g\,\epsilon^{abc} A^b_\mu F^c_{\mu\nu} = 0\,, \tag{1}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^b_\mu + g\epsilon^{abc} A^b_\mu A^c_\nu$ $(\mu,\nu = 0,1,2,3; a, b, c, = 1,2,3)$. *Here* A^a_μ are the elements of an arbitrary Lie algebra. Below we consider the case of the SU(2) algebra, so that the $\mathbf{A}_\mu$ may be identified with the vectors in a three-dimensional isotropic space. Equation (1) is consistent with a substitution[7]

$$A^a_0 = 0\,,\ \partial_i A^a_j = 0,\ A^a_i = O^a_i f^a\,,\ x = f^1\,,\ y = f^2\,,\ f^3 = 0 \tag{2}$$

(the O^a_i are constant orthogonal matrices, $O^a_i O^b_i = (1/g^2)\delta^{ab}$); this substitution reduces the equation to a system of ordinary differential equation,

$$\ddot{x} + x y^2 = 0\,,\quad \ddot{y} + x^2 y = 0\,;\quad \cdot \equiv \partial/\partial t\,. \tag{3}$$

This system has the obvious integral of motion

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + x^2y^2)\,. \tag{4}$$

We wish to emphasize that substitution (2) involves Eq. (1). Accordingly, from the fact that system (3) does not have an integral it follows immediately that the original system does not have a complete set of integrals, i.e., is nonintegrable.

That there is no integral of motion other than (4) can be checked with a computer. Let us outline the method which we used.

A fixed value of the integral determines a three-dimensional hypersurface in the four-dimensional space $(x, \dot{x}, y, \dot{y})$. We construct a Poincaré cross section of this space.[7] We cut it with a half-plane, e.g., $y = 0$, $\dot{y} > 0$. We assume that system (3) has an additional integral I_1. We fix its value. The intersection of the three-dimensional hypersurfaces defined by $E =$ const and $I_1 =$ const is a two-dimensional surface. In turn, the intersection of this surface with the secant half-plane is a closed curve (which lies in the secant half-plane). Corresponding to different values of I_1 at a fixed value of E are different closed curves. If the values of both integrals are fixed, all the trajectories of the system intersect the half-plane along the same curve. A periodic solution specifies closed curves in the phase space. These trajectories "pierce" the half-plane at a finite number of points N. A trajectory which begins from a point A_0 in the half-plane returns to A_0 after N intersections (or, equivalently, after the period of the solution, T). The point A_0 is thus a fixed point of the corresponding mapping of the half-plane into itself.

After linearization near a closed trajectory, system (3) is characterized by a matrix A^T which maps the vicinity of the fixed point into itself over the period T of the periodic solution.[8] The eigenvalues of this matrix, a monodromy matrix, determine the behavior of points near the fixed point. By virtue of the Liouville theorem, the existence of integral (4) implies $\det A^T = 1$. The product of the eigenvalues of the matrix is unity: $\lambda_1 \lambda_2 = 1$. We are interested in the case of real eigenvalues; in this case the point

A_0 is hyperbolic or unstable. A pair of curves, "stable and unstable separatrices," pass through it. Along the stable separatrix, trajectories intersect the half-plane as $t \to \infty$, approaching the point A_0; along the unstable separatrix, they move away from this point (and show the opposite behavior as $t \to -\infty$). Near the hyperbolic point A_0 the separatrices are determined by the eigenvectors of the monodromy matrix.

We mentioned above that if an additional integral existed all the curves in the half-plane would be closed. Consequently, the unstable and stable separatrices would transform into each other, forming a closed curve. If, on the other hand, the separatrices intersected at a nonzero angle (transversely), the system would have no additional integral.[7] The separatrices would intersect each other an infinite number of times, oscillating strongly, and coming arbitrarily close to the fixed point but they would never reach it, and the separatrices would not close.

In order to resolve the question of whether there is an additional integral it is thus necessary to find unstable solutions of system (3) and to study the behavior of the separatrix in the half-plane.

3. System (3) has the trivial periodic solutions

$$x = y = +F, \quad x = -y = -F, \tag{5}$$

where $F = cn(t\, 1/\sqrt{2})$ is the Jacobi elliptic cosine. In the half-plane with the coordinates $(x, \dot{x})$, the solutions in ($5 \pm$) have the fixed points $A_1(0;1/\sqrt{2})$ and $A_2(0; -1/\sqrt{2})$, respectively. The monodromy matrix A^T is the same near these points; its eigenvalues are real, $\lambda_1 = 129.647\,014$ and $\lambda_2 = 1/\lambda_1$. The points A_1 and A_2 are thus hyperbolic, and solutions ($5 \pm$) are unstable. Figure 1 shows separatrices constructed with the assistance of a computer; they intersect transversely and do not coincide. The angle at which the separatrices intersect at the point B (1.424 923 827 847;0) in the plane per-

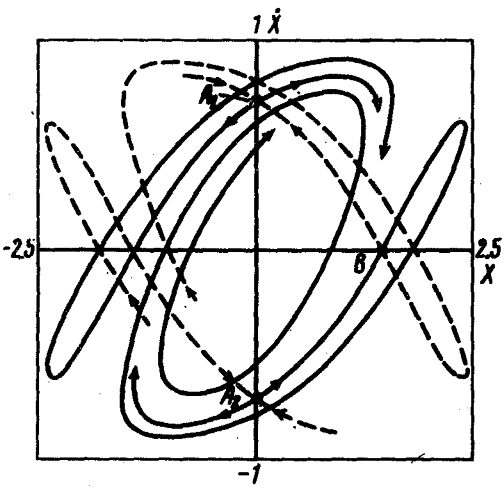

FIG. 1. Intersection of the phase space with the half-plane $y = 0$, $\dot{y} > 0$. Solid curves—unstable separatrices of the points A_1 and A_2; dashed curves—stable separatrices; B—one point at which the separatrices intersect. $E = 1/2$.

pendicular to the trajectory at this point is about 72°. The fact that the separatrices intersect proves that system (3) does not have an additional integral and proves that the original system of Yang-Mills equations, (1), does not have a complete set of integrals.

Ziglin[9] has proved a theorem stating that system (3) does not have an additional integral if it is assumed to be analytically continuable onto a band of finite width in the plane of the complex variable. This assumption is a strong one; it imposes serious restrictions on the form of the integral, so that the question is not resolved. In contrast, the intersection of separatrices observed by us forbids the existence of an additional integral without imposing restrictions of any sort on the physical nature of this integral.

We wish to thank E. B. Bogomol'nyĭ and Ya. G. Sinaĭ for active discussion of this study through its many stages. We also thank A. A. Belavin, V. E. Zakharov, S. P. Novikov, and É. I. Rashba for useful discussions.

[1]A. A. Belavin and V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 603 (1977) [JETP Lett. **25**, 567 (1977)].
[2]M. Atiyah and R. Ward, Commun, Math. Phys. **55**, 117 (1977);V. E. Zakharov, M. F. Ivanov, and L. N. Shur, Pis'ma Zh. Eksp. Teor. Fiz. **30**, 39 (1979) [JETP Lett. **30**, 34 (1979)].
[3]M. Atiyah, N. Hitchin, V. Drinfeld, and Yu. Manin, Oxford Preprint, 1978.
[4]A. M. Polyakov, Nucl. Phys. **B164**, 1971 (1980).
[5]H. J. Korsch, Phys. Lett. **90A**, 113 (1982).
[6]G. Z. Baseyan, S. G. Matinyan, and G. K. Savvidi, Pis'ma Zh. Eksp. Teor. Fiz. **29**, 641 (1979) [JETP Lett. **29**, 587 (1979)].
[7]H. Poincaré, Selected Works (Russ. transl., Vol. 2, Ch. XXXIII, Nauka, Moscow, 1972).
[8]V. I. Arnol'd, Obyknovennye differentsial'nye uravneniya (Ordinary Differential Equations), Nauka, Moscow, 1971.
[9]S. L. Ziglin, Funktsional'nyĭ analiz (Functional Analysis), No. 3, 30 (1982).

Translated by Dave Parsons
Edited by S. J. Amoretty