Hamiltonian formulation of the second-order drift equations of motion

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An expression is obtained for the Hamiltonian describing drift motion up to terms quadratic in the Larmor radius.

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In many problems arising in high-temperature plasma physics related to particle confinement in traps and in astrophysical applications, it is necessary to study the motion of a charged particle in a fixed electromagnetic field. The solution of this problem is greatly simplified for a field varying slowly in space and time, which is important in practice, if the drifting approximation is used.[1,2] The main small parameter of the drift theory (for simplicity, below we shall speak about stationary fields and nonrelativistic motion) is the ratio $r_L/l \equiv \epsilon$, where $r_L = mvc/eB$ is the Larmor radius of the particle, while l is the characteristic scale of variation of the fields. The first approximation with respect to the parameter ϵ is usually used, i.e., the drift velocity

$v_D \sim \epsilon v$ and corrections to the longitudinal velocity of the same order are included.

In some problems the accuracy of the first approximation of the drift theory is inadequate (for example, when calculating the coefficient of stochastic diffusion in long magnetic traps.[3]) In this paper we obtain expressions for second-order drift in curvilinear coordinates ξ^1, ξ^2, ξ^3, which are naturally related to the magnetic field by the expression

$$\mathbf{B} = [\nabla\xi^1 \times \nabla\xi^2]. \tag{1}$$

The corresponding equations of motion have a Hamiltonian form.

Although it has been known for sometime that the drift motion permits a Hamiltonian formulation,[4,5] this fact was usually ignored in the traditional approach. As a result, for example, the conservation of the longitudinal adiabatic invariant, which in the Hamiltonian approach follows from the general theorems of mechanics, required special proof.[1,2] The simplicity of the Hamiltonian description is also made evident in comparing our result (especially in the case of a potential magnetic field) with the results in Ref. 6, wherein an expression is presented for second-order drift in vector notation. We note another recent paper,[7] in which a drift Hamiltonian was also constructed, but up to terms $\sim\epsilon$.

We shall choose the curvilinear coordinates ξ^i as the generalized coordinates and write the Hamiltonian of a particle in a magnetic field:

$$H(\xi^i, p_i) = \frac{1}{2m} g^{ik}(\xi^1, \xi^2, \xi^3)\left[p_i - \frac{e}{c}A_i(\xi^1, \xi^2, \xi^3)\right]\left[p_k - \frac{e}{c}A_k(\xi^1, \xi^2, \xi^3)\right], \tag{2}$$

where g^{ik} represents the contravariant components of the metric tensor, p_i are the generalized momenta, and A_i are the covariant components of the vector potential, for which we choose the set $A_i = (0, \xi^1, 0)$. We shall make a canonical transformation from the variables p_i, ξ^i to the new variables $J_\perp, \varphi$; P, Q; $p_\parallel, s$ with the help of the generating function $F(\xi^1, \varphi; \xi^2, Q; \xi^3, p_\parallel)$:

$$F = \frac{e}{2cg^{22}}(B \operatorname{ctg}\varphi - g^{12})\left(\xi^2 - Q - \frac{c}{e}\frac{g_{13}}{g_{33}}p_\parallel\right)^2 + \frac{e}{c}\xi^1(\xi^2 - Q) + p_\parallel\frac{g_{23}}{g_{33}}(\xi^2 - Q) + p_\parallel \xi^3,$$

where $B(\xi^1, \xi^2, \xi^3)$ is the modulus of the magnetic field. In a homogeneous magnetic field $B = \text{const}$, the metric coefficients are constants, and for the new variables the following expressions are valid:

$$J_\perp = \frac{cm^2}{2eB} v_\perp^2, \qquad \varphi = \arccos\frac{\mathbf{v}_\perp \cdot \mathbf{a}^2}{v_\perp\sqrt{g^{22}}};$$

$$P = p_2, \qquad Q = \xi^2 - \frac{c}{e}p_1; \tag{3}$$

$$p_\parallel = p_3, \qquad s = \xi^3 + \frac{c}{e}\frac{g_{23}}{g_{33}}p_1 - \frac{c}{e}\frac{g_{13}}{g_{33}}\left(p_2 - \frac{e}{c}\xi^1 - \frac{g_{23}}{g_{33}}p_3\right),$$

where $\mathbf{a}^2 = \nabla\xi^2$, and $\mathbf{v}_\perp$ is the velocity component perpendicular to the magnetic field. As is evident, $J_\perp$ differs by only the factor cm/e from the magnetic moment of the particle φ is the Larmor phase measured from the direction $\mathbf{a}^2$, while $p_\parallel$ is the component of the particle momentum along the magnetic field. The remaining three variables determine the position of the particle up to r_L:

$$\xi^1 = \frac{c}{e}P + O(r_L), \quad \xi^2 = Q + O(r_L), \; \xi^3 = s + O(r_L). \tag{4}$$

In an inhomogeneous magnetic field, when calculating the partial derivatives of F with respect to ξ^i, it is necessary to differentiate the components of the metric tensor. As a result, additional small terms $\sim\epsilon$ will appear in relations (3). Expressing the old variables in terms of the new ones and substituting them into (2), we obtain a Hamiltonian in the new variables, which must then be averaged with respect to φ.[8] Omitting the intermediate calculations, we write out the drift Hamiltonian in the form $H = H_0 + H_1$, where $H_1 \sim \epsilon H_0$:

$$H_0 = \frac{eB}{mc} J_\perp + \frac{p_\parallel^2}{2m}$$

$$H_1 = \frac{B^2}{m} J_\perp p_\parallel \left[\frac{\partial}{\partial \xi^1}\left(\frac{g_{23}}{B}\right) - \frac{\partial}{\partial \xi^2}\left(\frac{g_{13}}{B}\right) + (g_{23})^2 \frac{\partial}{\partial \xi^3}\left(\frac{g_{13}}{B g_{23}}\right) - \frac{g^{22}}{2B^3}\frac{\partial}{\partial \xi^3}\left(\frac{g^{12}}{g^{22}}\right)\right] + \frac{c}{me} p_\parallel^3 g_{13} \frac{\partial g_{23}}{\partial \xi^3}. \tag{5}$$

(For simplicity, it is assumed here that $g_{33} \equiv 1$, which corresponds to identifying ξ^3 with the arc length along the force line.) The ξ^i dependent metric coefficients and the function B must be expressed in terms of the new variables according to Eqs. (4) [after dropping the terms $\sim O(r_L)$ in them].

It is remarkable that the term H_0 already describes first-order drift (centrifugal and gradient). Equations for $\dot{P}$, $\dot{Q}$, $\dot{p}_\parallel$, valid up to terms $\sim O(\epsilon^3)$ ($J_\perp$ in this approximation is an integral of the motion), follow from the Hamiltonian $H_0 + H_1$. The expression for $\dot{s} = \partial H/\partial p_\parallel$, has lower accuracy $O(\epsilon^2)$, which, of course, is not very important, since in many applications it is often sufficient to use only the zeroth-order approximation for $\dot{s}$: $\dot{s} = \partial H_0/\partial p_\parallel = p_\parallel/m$. We note the considerable simplification of the Hamiltonian H_1 in a potential field, where it is possible to choose a system of coordinates such that $g_{13} = g_{23} = 0$ (identifying ξ^3 with the magnetic potential).

In conclusion, we shall present one more expression for the contribution to the Hamiltonian related to the electric potential $\varphi(\xi^1, \xi^2, \xi^3)$. In this case, the term $e\varphi\,[(c/e)P, Q, s]$ is added to H_0 and the term

$$c p_\parallel \left(-\frac{\partial \varphi}{\partial \xi^1} g_{23} + \frac{\partial \varphi}{\partial \xi^2} g_{13} - \frac{\partial \varphi}{\partial \xi^3} g_{13} g_{23} \right)$$

is added to H_1.

1)If the system of coordinates is chosen so that $g_{13} = g_{23} = 0$, then it can be shown that the set $(cP/e, Q, s)$ gives the coordinates of the center of the Larmor circle, although in general this is not the case.

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