Mechanism by which spatially homogeneous Yang-Mills fields become stochastic

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A mechanical system with a nonzero angular momentum, corresponding to spatially homogeneous Yang-Mills fields, is analyzed. Numerical simulations have been carried out. A mechanism by which the fields become stochastic is found.

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It was shown in Ref. 1 that an analysis of spatially homogeneous solutions of the classical Yang-Mills equations reduces to an analysis of mechanical systems. It was shown in Refs. 2 and 3 that a system with a Hamiltonian $H=(1/2)(\dot{x}^2+\dot{y}^2+x^2y^2)$, which describes the interaction of two color degrees of freedom, is stochastic. In the general case,[4] the problem of spatially homogeneous Yang-Mills fields can be reduced to an analysis of a mechanical system with a Hamiltonian

$$\epsilon=\frac{1}{2}(\dot{r}^2+\dot{R}^2)+\frac{\mu^2}{32}\left(\frac{1}{r^2}+\frac{1}{R^2}\right)+\frac{1}{4}(r^2-R^2)^2, \tag{1}$$

where μ is the analog of the angular momentum. The system studied in Refs. 2 and 3 corresponds to the value $\mu=0$.

The gauge transformations $r\to(\mu/4)^{1/3}r$, $R\to(\mu/4)^{1/3}R$, $t\to(\mu/4)^{-1/3}t$ put Hamiltonian (1) in the form

$$E=\epsilon(4/\mu)^{4/3}=1/2(\dot{r}^2+\dot{R}^2)+\frac{1}{2}\left(\frac{1}{r^2}+\frac{1}{R^2}\right)+\frac{1}{2}(r^2-R^2)^2 \tag{2}$$

with the equations of motion

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$$\ddot{r} = \frac{1}{r^3} - r(r^2 - R^2); \qquad \ddot{R} = \frac{1}{R^3} + R(r^2 - R^2). \tag{3}$$

In terms of the variables $x = r - R$, $y = r + R$, Hamiltonians (2) and system (3) become

$$2E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{4}{(y+x)^2} + \frac{4}{(y-x)^2} + \frac{x^2y^2}{2}, \tag{4}$$

$$\ddot{x} = \frac{8}{(y+x)^3} - \frac{8}{(y-x)^3} - xy^2, \tag{5a}$$

$$\ddot{y} = \frac{8}{(y+x)^3} + \frac{8}{(y-x)^3} - yx^2. \tag{5b}$$

The equipotential lines of system (5) can be parametrized in the form

$$x = \pm\left[-2\sqrt{s} - \frac{1}{s} + \sqrt{\frac{1}{s^2} + 4s + 4U}\right]^{1/2}; \quad y = \left[2\sqrt{s} - \frac{1}{s} + \sqrt{\frac{1}{s^2} + 4s + 4U}\right]^{1/2}, \tag{6}$$

where $s \geqslant 1/U^2$. We note that the motion of a particle with an energy E occurs within the region bounded by line (6), with a potential energy $U = E$. This region is divided into regions of negative and positive values of the y component of the force in (5b). We call the latter a "channel."

Can a particle move off along a channel to infinity? Inside a channel at large values of y the motion along y is determined by $y(t) = [16/c + ct^2]^{1/2}$. The motion along x is determined by the equation $\ddot{x} + (16/c + ct^2)x = 0$. At $t \geqslant 4/c$ the amplitude of the x oscillations falls off in proportion to $t^{-1/2}$, while the channel width decreases in accordance with $y^{-2} \sim t^{-2}$. A particle thus cannot go off to infinity along a channel.

In the region $x \ll y$, system (5) is described by the approximate equations

$$\ddot{x} + xy^2 = 0; \qquad \ddot{y} - \frac{16}{y^3} + x^2y = 0. \tag{7}$$

Since the coordinate x oscillates vigorously, it is convenient to introduce the parametrization[5]

$$x = \alpha\,\rho(t)^{-1/2} \sin\left(\int_0^t \rho(z)\,dz + \varphi\right). \tag{8}$$

System (7) then becomes

$$\sqrt{\rho}\frac{d^2}{dt^2}\left(\frac{1}{\sqrt{\rho}}\right) + y^2 = \rho^2; \quad \ddot{y} = \frac{16}{y^3} - \alpha^2\frac{y}{\rho}\sin^2\left(\int_0^t \rho(z)\,dz + \varphi\right). \tag{9}$$

Within terms of the order of x^2/y^2, we find $\rho = y$ from the first of these equations. We see that the second term in the last equation in (8) may be regarded as a rapidly oscillating external force which gives rise to an effective potential well for the motion along y. Taking the time average of the term $\sin^2[\int_0^t \rho(z)dz + \varphi]$, we find that the potential well is described by $U = 8/y^2 + \alpha^2 y/2$. It is important to note that α is not a

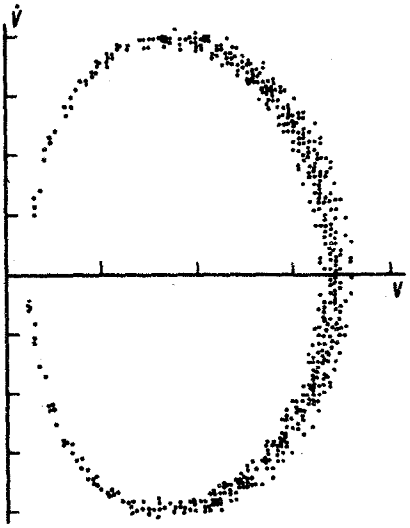

FIG. 1.

universal property of all the trajectories, it is instead determined by the initial conditions. We might note that the appearance of regions of stable y oscillations is analogous to the Kapitsa effect.[5] Motion at the bottom of the well corresponds to low energies ($E \leqslant 1$). At low energies, we might note, the entire accessible region of motion satisfies the condition $y \gg x$.

System (5) has been solved numerically. The motion of a particle was studied for various initial conditions and for several energies. The integral of motion in (4), we might note, causes the phase space to be three-dimensional. As the independent coordinates we chose $u = xy$, $v = \frac{1}{2}(x^2 + y^2)$, and $\dot{v}$. Figure 1 shows intersections of the phase trajectory of the particle with the energy $E = 1$ in the $u = 0$ plane. For this energy, system (5) is approximately integrable. The reason for the asymmetry in this

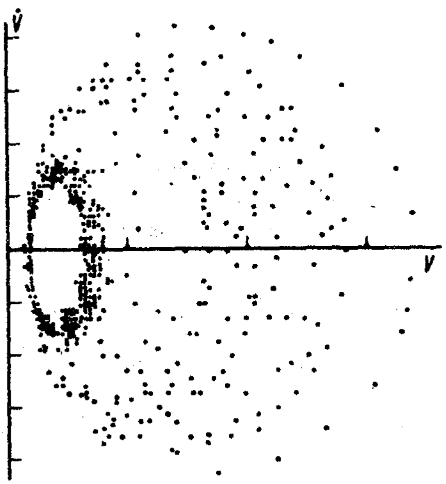

FIG. 2.

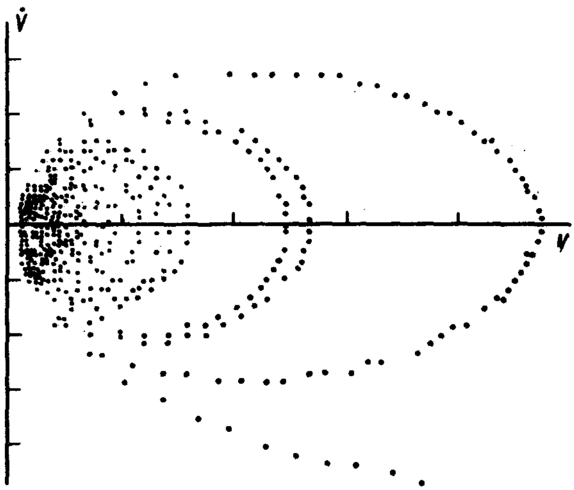

FIG. 3.

figure is that the right side of the effective well retains its shape only on the average. As E increases, the width of the oval curve increases, reaching its dimensions at $E = 1.3$–1.4. Figure 2 corresponds to the case $E = 1.4$. With a further increase in the energy, the phase trajectories exhibit a qualitative change in behavior. Ovals of different sizes appear, and the blurring of these ovals decreases. Figure 3 corresponds to the case $E = 3$. This behavior of the system can be understood on the basis of the following qualitative analysis. As E increases, the minimum value $y_{\min} = 2/\sqrt{E}$ of the coordinate y decreases, while $x_{\max}$ increases. Beginning at a certain E, the particle can enter the "root region" $x \sim y$, where the concept of an effective potential well breaks down. Here the particle experiences, on the average, a force with a positive y component, which tends to expel the particle from the root region. The subsequent motion depends strongly on the value of the parameter α. Depending on the complex motion in the root region, α can take on essentially random values. The singularity in the determination of the oscillation amplitude $y_{\max} \sim 2E/\alpha^2$ shows that $y_{\max}$ can take on arbitrarily large values.

The random redetermination of the parameter α in the root region may be thought of as a distinctive mechanism which renders the motion stochastic.

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