Quantization of relativistic Fermi-Bose systems with first- and second-order couplings by the compensating functional method

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The compensating functional method is used to obtain a universal expression for the *S* matrix of an arbitrary Fermi-Bose system with general couplings.

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In recent years, attempts to build supergravitational models that unify the strong, weak, electromagnetic, and gravitational interactions received special attention.

These models represent gauge systems with Fermi-Bose coupling. Fradkin[1] carried out quantization of an arbitrary Fermi-Bose system with first- and second-order couplings, and obtained a correct expression for the S matrix of such system under canonical gauge conditions. The problem of quantization of systems with coupling in the relativistic gauge was solved recently in a number of papers.[2-5]

While the expression in Ref. 1 for the S matrix for systems under the canonical gauge conditions has a universal form for all theories, it is not universal for systems under the relativistic gauge conditions and is tailored for each theory individually. These characteristics are determined by the open Lagrangian gauge algebra for the theories with a rank greater than unity.

Although the expression in Ref. 5 for the S matrix with a certain specific rank is more adequate and convenient for constructing a diagram in the theory, a universal explanation of an arbitrary general theory in the relativistic gauge is of great interest.

Batalin and Fradkin[6] recently obtained a universal expression for the S matrix of an arbitrary Bose system with first-order coupling in the relativistic gauge. The purpose of this paper is to obtain a universal expression for the S matrix for the general Fermi-Bose systems with first- and second-order couplings.

A dynamic system with coupling is described by the following interaction Hamiltonian:

$$S = \int (p_i \dot{q}^i - H(p, q) - T_\mu (p, q)\lambda^\mu)\, dt \,, \tag{1}$$

where q^i and p_i are coordinates of the phase space, H is the Hamiltonian of the system, λ^μ are Lagrangian multipliers, and T_μ is the first-order coupling in the involution:

$$\{ T_\alpha , T_\beta \}_D = T_\gamma U^{\gamma}_{\alpha\beta} \,, \tag{2}$$

$$\{ H_o , T_\alpha \}_D = T_\beta V^{\beta}_{\alpha} \,. \tag{3}$$

We shall define the Dirac brackets[1]

$$\{ A, B\}_D = \{ A, B\} - \{ A, \chi_k \}\{ \chi_k, \chi_s \}^{-1}\{ \chi_s , B \} , \tag{4}$$

where χ_k is the second-order coupling and $\{....\}$ are Poisson brackets for the general case of the Fermi-Bose particles[1]:

$$\{ A, B \} = \frac{\partial^r A}{\partial q^a} \frac{\partial^e B}{\partial p_a} - (-1)^{n_A n_B} \frac{\partial^r B}{\partial q^a} \frac{\partial^e A}{\partial p_a} \,. \tag{5}$$

Or, introducing the generalized variable g^a,

$$g^a = \left(\frac{q^i}{p_i}\right).$$

We shall transform Eq. (5) as follows:

$$\{A, B\} = \frac{\partial^r A}{\partial g^a} \epsilon^{ab}_{(B)} \frac{\partial^r B}{\partial g^b} \tag{6}$$

$$\epsilon^{ab}_{(B)} = \begin{pmatrix} 0 & \delta^{ab}(-1)^{n_a(n_B+1)} \\ -\delta^{ab}(-1)^{n_a n_B} & 0 \end{pmatrix}. \tag{7}$$

Equation (1) is invariant with respect to gauge transformations

$$\delta g^a = R^a_\alpha(g)\theta^\alpha, \tag{8}$$

$$\delta\lambda^\mu = \hat{R}^\mu_\alpha(g, \lambda)\theta^\alpha, \tag{9}$$

where

$$R^a_\alpha(g) = \{g^a, T_\alpha\}_D, \tag{10}$$

$$\hat{R}^\mu_\alpha(g, \lambda) = \delta^\mu_\alpha \frac{d}{dt} + U^\mu_{\alpha\beta}\lambda^\beta - V^\mu_\alpha. \tag{11}$$

We introduce the gauge conditions using the functional:

$$\Psi^\alpha(t/g, x\lambda) \tag{12}$$

The method[6] essentially involves derivation of the gauge-invariant and unitary expression for the generating functional in the generalized gauge [Eq. (12)], which depends on the parameter "x"; this dependence is chosen in such a way that for $x = 0$ we have a canonical gauge and for $x = 1$, the gauge acquires the desired relativistic form. We shall express the generating functional for such a system with coupling in the generalized gauge form [Eq. (12)]:

$$Z = \int [d\mu] \exp\{iS\} \delta(\Psi(g/x\lambda)) \Delta(x/g, \lambda) \tag{13}$$

$$[d\mu] = [dg][d\lambda] M, \tag{14}$$

$$M = \delta(\chi_k) \exp\left[\frac{1}{2} Tr_\pm \ln Q_{ab}\right], \tag{15}$$

$$Tr_\pm \bar{Q}_{ab} = \sum_a \epsilon_a \bar{Q}_{aa}; \quad Q_{ab} = \{\chi_a, \chi_b\}; \quad \epsilon_a = \begin{cases} 1, & a - \text{Bose} \\ -1, & a - \text{Fermi}, \end{cases}$$

where $\Delta(x/g, \lambda)$ is the so-called compensating functional chosen in such a way that the S matrix is independent of gauge (independent of the parameter x). We note that the generating functional [Eq. (13)] is constructed in such a way that at $x = 0$ it becomes a well-known expression for the generating functional of the Fermi-Bose systems with first- and second-order couplings in the canonical gauges[1]; hence it guarantees its unitarity for all values of x. Since s is gauge invariant [i.e., the gauge transformations (8) and (9) do not change the shape of s], we shall perform a gauge transformation of Eqs. (8) and (9) that transforms the $\Psi^\alpha(g/x\lambda)$ gauge into $\Psi^\alpha[g/(x+dx)\lambda] = \Psi^\alpha(g/x\lambda) + \delta\Psi^\alpha$ gauge. It can easily be verified that such an increment $\delta\Psi^\alpha = \frac{d^r\Psi^\alpha}{dx}dx$ of the initial gauge is induced by the gauge transformations (8) and (9) with the parameter θ^α:

$$\theta^a = F^a(t;\ x/g,\lambda)\,dx, \tag{16}$$

where

$$F^a(t;x/g,\lambda) = \int D^a_\beta(t,t';x/g,\lambda)\,\frac{d^r\Psi^\beta(t'/g,x\lambda)}{dx}. \tag{17}$$

The D function can be determined from the condition

$$\int dt' D^a_\beta(t',t'';x/g,\lambda)\,D^{-1\beta}_\gamma(t',t;x/g,\lambda) = \delta^a_\gamma\,\delta(t''-t), \tag{18}$$

where

$$D^{-1a}_\beta(t',t;x/g,\lambda) = \{\Psi^a(t'),T_\beta(t)\}$$

$$+\ \frac{\delta^r\Psi^a(t/g,x\lambda)}{\delta\lambda^\mu(t')}\,\hat{R}^\mu_\beta(g/t'),\lambda(t')). \tag{19}$$

As was mentioned above, the compensating functional $\Delta(x/g,\lambda)$ is chosen so that the generating functional is independent of the parameter x; taking into account the variation Δ and the dimensions M, we obtain the following equation for Δ [1]:

$$\frac{\partial^r\Delta}{\partial x} = [\hat{\Omega} + J(x/g,\lambda)]\Delta, \tag{20}$$

where

$$\hat{\Omega}\Delta = \int dt\left[\{\Delta, T_a(t)\}_D + \frac{\delta^n\Delta}{\delta\lambda^\mu}\hat{R}^\mu_a(g(t),\lambda(t))\right]F^a(t;x/g,\lambda), \tag{21}$$

$$J(x/g,\lambda) = \int dt\left[-\{g^a(t),\delta^F g^a(t)\}_D + (-1)^{n\mu}\frac{\delta^r(\delta^F\lambda^\mu(t))}{\delta\lambda^\mu(t)}\right]. \tag{22}$$

The unitarity of the S matrix is guaranteed by the initial conditions of Δ:

$$\Delta(x=0) = \Delta_0(g) = \mathrm{Det}_\pm D_0^{-1}(t,t'/g),$$

where "$\pm$" has the same meaning as that in Eq. (15). In this case, for the canonical gauge conditions, the expression for Z coincides with the expression for the S matrix derived by Fradkin.[1] According to Ref. 6, we express Δ in the form

$$\Delta = \Delta_1 \mathrm{Det}_{\pm}\, D^{-1} \tag{23}$$

Thus, Eq. (20) becomes

$$\frac{\partial^r \Delta_1}{\partial x} = [\hat{\Omega} + J_1(x/g, \lambda)]\Delta_1 , \tag{24}$$

where

$$J_1(x/g, \lambda)$$

$$= \int dt\, dt' D^{\beta}_{\gamma}(t; t'; x/g, \lambda)[\{\Psi^{\gamma}(t'/g, x\lambda), U^{\nu}_{\beta a}(g(t))\}_D T_{\nu}(g(t))(-1)^{n_\nu n_{\nu a\beta}}$$

$$+ \frac{\delta^r \Psi^{\gamma}(t'/g, x\lambda)}{\delta \lambda^{\mu}(t)} (- \dot{U}^{\mu}_{\beta a}(g(t)) + \{ U^{\mu}_{\beta a}(g(t)),\ H(g(t)$$

$$+ T_{\nu}(g(t))\, \lambda^{\nu}(t)\}_D + B^{\mu}_{a\beta\nu}(g(t)\lambda^{\nu}(t)(-1)^{n_a n_{\nu\beta}}$$

$$+ B^{\mu}_{a\beta}(g(t))\,(-1)^{n_a n_\beta})]\, F^a(t; x/g, \lambda)(-1)^{n_\beta}; \tag{25}$$

$$B^{\mu}_{a\beta\nu}(g) = (-1)^{n_a n_\nu}\{U^{\mu}_{a\beta}, T_\nu\}_D + (-1)^{n_\nu n_\beta}\{U^{\mu}_{\nu a}, T_\beta\}_D$$

$$+ (-1)^{n_a n_\beta}\{U^{\mu}_{\beta\nu}, T_a\}_D - U^{\mu}_{\nu\delta} U^{\delta}_{a\beta}(-1)^{n_\beta n_\nu} - U^{\mu}_{\beta\delta}\, U^{\delta}_{\nu a}(-1)^{n_a n_\beta}$$

$$- U^{\mu}_{a\delta}\, U^{\delta}_{\beta\nu}(-1)^{n_a n_\nu}; \tag{26}$$

$$B^{\mu}_{a\beta}(g) = \{V^{\mu}_a, T_\beta\}_D - \{V^{\mu}_\beta, T_a\}_D(-1)^{n_a n_\beta} + \{U^{\mu}_{a\beta}, H\}_D$$

$$+ U^{\mu}_{a\delta}\, V^{\delta}_{\beta} - V^{\mu}_{\delta}\, U^{\delta}_{a\beta} - (-1)^{n_a n_\beta} U^{\mu}_{\beta\delta}\, V^{\delta}_{a}. \tag{27}$$

We obtain from the generalized Jacobi identities the condition for the functions (26) and (27):

$$\begin{cases} T_\mu B^{\mu}_{a\beta\gamma} = 0 & (28) \\ T_\mu B^{\mu}_{a\beta} = 0 & (29) \end{cases}$$

Thus, the initial conditions for Δ_1 are: $\Delta_1(x=0)=1$. The equations for the characteristics (24) can be written as follows:

$$-\frac{d}{dx'}\begin{pmatrix}\bar{g}^a\\ \bar{\lambda}^\mu\end{pmatrix}=\begin{pmatrix}R^a_{\ a}(\bar{g})\\ \hat{R}^\mu_a(\bar{g},\bar{\lambda})\end{pmatrix}F^a(t;\,x'/g,\lambda). \qquad (30)$$

With the initial conditions:

$$\bar{g}^a(x'=x)=g^a(t),$$
$$\bar{\lambda}^\mu(x'=x)=\lambda^\mu(t). \qquad (31)$$

The solution of Eq. (24) with the initial conditions (29) is

$$\Delta_1(x/g,\lambda)=\exp\{\int_0^x J_1(x'/\bar{g}(x'),\bar{\lambda}(x'))\,dx'\}. \qquad (32)$$

We take into the account the fact $x=1$ corresponds to the relativistic gauges and obtain Batalin-Fradkin expression for the generating functional of the general theory with the Fermi-Bose first- and second-order couplings:

$$Z=\int[d\mu]\exp\{iS(g,\lambda)\}\delta(\Psi(g,\lambda))\,\Delta_1(1/g,\lambda)\,\mathrm{Det}_\pm D^{-1}, \qquad (33)$$

where J_1 is determined from the relations (25)–(27), Δ_1 is determined from Eq. (32), and $\bar{g}$ and $\bar{\lambda}$ are determined by solving the characteristics equations [Eq. (30)] with the initial conditions (31).

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[1)]The compensating functional that satisfies a relationship such as Eq. (20) was used by Batalin[7] in formulating a gauge-invariant S matrix for the massive Yang-Mills field.

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