Interaction of instantons and anti-instantons in a nonlinear O(3)σ model and a certain exactly solvable fermion theory

A. P. Bukhvostov and L. N. Lipatov
B. P. Konstantinov Institute of Nuclear Physics, USSR Academy of Sciences

(Submitted 24 November 1979)
Pis'ma Zh. Eksp. Teor. Fiz. **31**, No. 2, 138–142 (20 January 1980)

A two-component, two-dimensional fermion model, which has an exact solution, is proposed on the basis of calculation of the energy of interaction of the instantons and anti-instantons in the O(3)σ model.

PACS numbers: 11.10.Lm, 11.30.Na

1. Of greatest interest among all the two-dimensional Euclidean models of quantum field theory is the nonlinear σ model with the action

$$S = \frac{1}{2f} \int d^2x \, (\partial_\mu n(x))^2, \qquad n^2(x) = 1\,, \tag{1}$$

where $n(x)$ is a three-dimensional unit vector. This interest is due to the close analogy between the σ model and the Yang-Mills theory and, in particular, due to the presence in it of instantons.[1] If $n(x)$ is parametrized by using a complex function $w(x_1,x_2)$:

$$n_1 = \operatorname{Re} w, \quad n_2 = \operatorname{Im} w, \quad n_3 = \frac{|w|^2 - 1}{|w|^2 + 1}, \tag{2}$$

then the instanton—solution of the equation $\delta S = 0$ with a topological charge $n > 0$—has the form[1,2]:

$$w = h \prod_{i=1}^{n} \frac{z - a_i}{y - b_i}, \qquad z = x_1 + ix_2, \tag{3}$$

where h, a_i, and b_i are arbitrary complex parameters. The solution corresponding to the anti-instanton with a topological charge $(-n)$ is obtained from Eq. (3) but substituting $z \to z^*$.

Fateev *et al.*[2] calculated the contribution to the statistical sum $Z = \int Dn \exp(-S)$ from all the configurations (3) with allowance for quantum fluctuations around the classical solutions. The obtained expression corresponds to the statistical sum of the Coulomb gas at a certain temperature; the parameters a_i and b_i are coordinates of the positively and negatively charged particles. At the given temperature the statistical sum Z may also be written in terms of the noninteracting massive fermions. In this paper, on the basis of calculation of the contribution to the statistical sum from the configurations corresponding to weakly interacting instantons and anti-instantons, we obtain a certain model which has an exact solution.

2. We select for the $w(x_1,x_2)$ function, which describes the interacting instantons and anti-instantons, the following expression [in analogy with Eq. (3)]:

$$w = h \prod_{i=1}^{n_1} \frac{z - a_i}{z - b_i} \prod_{j=1}^{n_2} \frac{z^* - c_j^*}{z^* - d_j^*}. \tag{4}$$

Equation (4) gives an approximate solution of the classical equations $\delta S = 0$, if the interaction of the instantons with the anti-instantons is small. This holds for such configurations in which the instantons and anti-instantons differ in size (for example, $(|a_{i_1} - b_{i_2}| \ll |c_{j_1} - d_{j_2}|)$ or are separated far apart $(|a_{i_1} - b_{i_2}| \sim |c_{j_1} - d_{j_2}| \ll |a_{i_1} - c_{j_1}|)$. The interaction energy S_{int} is defined as the difference between the action (1) calculated in the configuration (4) and the sum of the values corresponding to the noninteracting instantons and anti-instantons; in the limit $S_{\text{int}} \ll S$ it is

$$S_{int} = S - \frac{4\pi}{f}(n_1 + n_2) = \sum_{ij} \frac{16\pi |w_{ij}|^2}{f(1 + |w_{ij}|^2)^2} \operatorname{Re} \frac{(a_i - b_i)(c_j - d_j)}{(a_i - c_j)(b_i - d_j)}. \tag{5}$$

Here w_{ij} is the value of the function (4) in the region that separates the group of instantons to which an instanton with the parameters a_i and b_i belongs, from the group of anti-instantons to which an anti-instanton with the parameters c_j and d_j belongs. [In the limit of smallness of the interaction such region of slow variation of the $w(z)$ function is applicable to any i and j whose contribution to the sum (5) is nonnegligible.] Note that expression (5) is conformally invariant and also has a certain symmetry property, which is a consequence of the invariance of Eq. (1) with respect to the global rotations of the vector n.

The quantum fluctuations near the approximate solution of Eq. (4) within the limit of the weakly interacting instantons and anti-instantons are calculated exactly the same way as in Ref. 2, which gives the expression for the statistical sum of the Coulomb gas consisting of two types of particles. The interaction of different type of particles is described by the factor $\exp(-S_{\mathrm{int}})$ [see Eq. (5)].

3. We shall attempt to construct a fermion model which can correctly describe the interaction of instantons and anti-instantons at least at large distances. We can verify that if we write for the statistical sum the expression corresponding to two interacting fermions ψ_1 and ψ_2 with the Lagrangian

$$L = \sum_{r=1,2} \bar{\psi}_r (i\hat{\partial} - m)\psi_r - g(\bar{\psi}_1 \gamma_\mu \psi_1)(\bar{\psi}_2 \gamma_\mu \psi_2), \tag{6}$$

then in the lower-order perturbation theory with respect to the g constant the correction for the statistical sum coincides with the correction for interaction of the aforementioned Coulomb gas, with the only difference that the factor in front of Re in Eq. (5) is replaced by $-2g/\pi$. Thus, examining further the fermion theory with the Lagrangian (6), we expect to correctly describe only the interaction of the instantons and anti-instantons at large distances, where the indicated factor is independent of the parameters a, b, c, and d.

4. The field theory with the Lagrangian (6) allows an exact solution which can be obtained by using the methods developed for the massive Thirring model.[3] If we go over from the Lagrangian description to the Hamiltonian description and look for a solution of the Schrödinger equation $H|\phi\rangle = E|\phi\rangle$ in the form of a state with a definite number of fermions with a positive or a negative energy, then we can show that the wave function, which describes n fermions with momenta k_i and energy symbols σ_i, can be determined in the form of Bethe equation (see Ref. 4):

$$\chi_{a_1 \ldots a_n}(x_i, a_i)\big|_{x_1 < \ldots < x_n} = \sum_{\{i_1, \ldots i_n\}} (-1)^{P\{i_1, \ldots i_n\}}$$

$$\times A^{i_1 \ldots i_n}_{a_1 \ldots a_n} \prod_{r=1}^{n} f_{k_{i_r} \sigma_{i_r}}(x_r, a_r); \tag{7}$$

$$f_{k\sigma}(x, a) = e^{ikx} u_\sigma(a); \quad u_\sigma(a) = \begin{pmatrix} e^{-\frac{y}{2}} \\ \sigma e^{\frac{y}{2}} \end{pmatrix},$$

where $\alpha_i = 1, 2$ correspond to two possible types of fermions ("isospin"), $\{i_1, \ldots i_n\}$ denotes a certain permutation of numbers 1, 2...n, $P\{\ \}$ is the parity of this permutation, and y is the velocity of a particle: $K = \sigma m \sinh y$. The $A^{i_1 \cdots i_n}_{a_1 \cdots a_n}$ coefficients can be expressed in terms of the $A^{1\cdots n}_{a_1\cdots a_n}$ coefficient by using the formula:

$$A^{i_1\cdots i_n}_{a_1\cdots a_n} = S^{a'_{r+1}a'_r}_{a_r a_{r+1}}(y_r - y_{r+1})A^{i_1\cdots i_{r+1}i_r\cdots i_n}_{a_1\cdots a'_{r+1}a'_r\cdots a_n}, \tag{8}$$

where S is a two-particle S matrix for scattering of fermions in this theory:

$$S^{a'b'}_{ab}(y) = \delta_{aa'}\delta_{bb'}\delta_{ab} + \frac{1}{2}\epsilon_{cab}\epsilon_{ca'b'}\frac{\sinh\left(\frac{y-ig}{2}\right)}{\sinh\left(\frac{y+ig}{2}\right)} + \frac{1}{2}(\tau_x)_{ab}(\tau_x)_{a'b'}\frac{\cosh\left(\frac{y-ig}{2}\right)}{\cosh\left(\frac{y+ig}{2}\right)}. \tag{9}$$

The matrix (9) satisfies the factorization condition,[3] which ensures a compatibility of Eqs. (8) and makes it possible to find an exact solution of the model with the Lagrangian (6).

5. To construct a physical vacuum, we must find a solution of the Schrödinger equation with minimum energy. This formulation of the problem is correct only in the finite volume of the space L and in the presence of a cutoff Λ for the permissible velocities ($|\gamma| < \Lambda$). Specifically, we can require that the $\chi_{a_1\cdots a_n}(x_i, \alpha_i)$ function should satisfy the periodic boundary conditions of each of the coordinates x_i. To determine all the values of y_i and of the matrix $A^{1\cdots n}_{a_1\cdots a_n}$ in Eqs. (7), for which the periodic conditions are satisfied, we use the quantum method of the inverse problem.[6] We write only the final equations for determining the spectrum of the values $y_i (i = 1,\ldots n)$ and certain parameters v_r $(r = 1,\ldots l)$ whose number is equal to the number of type 2 fermions (the number of type 1 fermions is $n - l$):

$$e^{imL\,\mathrm{sh}\,y_j} = \prod_{r=1}^{e}\frac{\sinh\left(v_r - y_j + \frac{ig}{2}\right)}{\sinh\left(v_r - y_j - \frac{ig}{2}\right)};$$
$$\prod_{j=1}^{n}\frac{\sinh\left(v_s - y_j + \frac{ig}{2}\right)}{\sinh\left(v_s - y_j - \frac{ig}{2}\right)} = \prod_{\substack{r=1\\ r\neq s}}^{l}\frac{\sinh(v_s - v_r + ig)}{\sinh(v_s - v_r - ig)}. \tag{10}$$

Equations (7)–(10) allow us to find all solutions of the steady-state Schrödinger equation corresponding to the theory with the Lagrangian (6). The energy and momentum for these states are equal to the sum of the same values for the individual fermions.

In the continuous limit, it is convenient to introduce fermion density distributions with respect to the velocities y and parameters v: $\rho(y_i) = [L(y_i - y_{i-1})]^{-1}$, $\mu(v_r) = [L(v_r - v_{r-1})]^{-1}$. In the solution of Eqs. (10) for the state with the minimum energy (physical vacuum), we can see that $\rho(y)$ coincides with an analogous value for the noninteracting model ($g = 0$). Specifically, it follows from this that the rarefied-gas approximation, which is usually used in the Yang-Mills theory to determine the contribution of the instantons to the ground-state energy, may give the correct answer, if the interaction between the instantons is calculated exactly, and the indicated approximation is used only to calculate the interaction of instantons with anti-instantons. In the next paper we hope to discuss the mass spectrum produced in the model (6).

The authors thank A. A. Belavin, E. B. Bogomol'nyĭ, and A. S. Shvarts for useful discussions.

[1]A. A. Belavin and A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 503 (1975) [JETP Lett. **22**, 245 (1975)].
[2]V. A. Fateev, I. V. Frolov, and A. S. Shvarts, Nucl. Phys. **B154**, 121 (1979).
[3]H. Bergknoff and H. B. Thacker, Preprint Fermilab-Pub 78/61-THY, 1978; V. E. Korepin, Teor. Mat. Fiz. **40**, 316 (1979).
[4]F. A. Berein and V. N. Sushko, Zh. Eksp. Teor. Fiz. **48**, 1293 (1965) [Sov. Phys. JETP **21**, 865 (1965)].
[5]C. N. Yang, Phys. Lett. **19**, 1312 (1967); A. B. Zamolodchikov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 449 (1977) [JETP Lett. **25**, 468 (1977)].
[6]R. Y. Baxter, Ann. Phys. **70**, 193 (1972); E. K. Sklyanin, L. A. Takhtadzhyan, and L. D. Faddeev, Teor. Mat. Fiz. **40**, 194 (1979); A. A. Belavin, Phys. Lett. **B76**, 327 (1979).