Stability of power solitons in one-dimensional nonlinear systems

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(Submitted 25 December 1979)

Pis'ma Zh. Eksp. Teor. Fiz. **31**, No. 4, 213–217 (20 February 1980)

It is shown that the power solitons are absolutely unstable in certain one-dimensional, nonlinear systems.

PACS numbers: 02.30.Jr, 03.65.Ge

Lately, much attention has been devoted to solitons in studying nonlinear systems. It was shown recently[1–3] that a number of one-dimensional, nonlinear equations have soliton solutions that have a power-law asymptotic form. They are derived from the ordinary exponential solitons by specially selecting their parameters. We want to draw attention to the importance of the stability of power solitons. We show below the absolute instability of such solutions in some specific cases.

1. Let us examine a nonlinear Schrödinger equation

$$i\dot{\psi} + \psi'' - \psi + f\{|\psi|^2\}\psi = 0, \tag{1}$$

to which many nonlinear equations reduce in first approximation when they are solved asymptotically.[4] Seeking solutions of the form $\psi = u(x)\exp(i\omega t)$, we obtain the following equation for $u(x)$:

$$u'' - (1+\omega)u + f(u^2)u = 0. \tag{2}$$

In studying the soliton solutions, we generally limit ourselves to the first term in the $f(z)$ expansion: $f(z) = \alpha z$. Thus, at $\alpha > 0$ Eq. (2) has a single solution for all $\omega > -1$. When the oscillation amplitudes are small and the coefficient of the second term of the $f(z)$ expansion is anomalously large, two terms must be left in it. The soliton in this case can exist even when $\alpha < 0$. If the second term is positive, then

$$u'' - (1+\omega)u - |\alpha|u^3 + \beta^2 u^5 = 0. \tag{3}$$

For arbitrary $\omega > -1$ the soliton solution has the form

$$u = \frac{\epsilon}{\sqrt{|\alpha|}}\left\{\sqrt{1 + \frac{4}{3}\left(\frac{\beta\epsilon}{\alpha}\right)^2}\,\mathrm{ch}\,\epsilon x - 1\right\}^{-1/2}; \qquad \epsilon = 2\sqrt{1+\omega}. \tag{4}$$

When $\epsilon \to 0$ $(\omega \to -1)$ the solution (4) goes over to a power soliton

$$u = \sqrt{\frac{2}{|\alpha|}}\left\{x^2 + \frac{4}{3}\left(\frac{\beta}{\alpha}\right)^2\right\}^{-1/2}. \tag{5}$$

It is easy to demonstrate that the solutions (4) and (5) are unstable in the sense that the small corrections to them increase exponentially. It was shown[5] that the instability of solitons in the nonlinear Schrödinger equation is due to the sign of the derivative $dI/d\omega$, where $I = \int_{-\infty}^{\infty} u^2 dx$. The solitons are stable when $dI/d\omega > 0$ and are unstable when $dI/d\omega < 0$. In our case,

$$I(\omega) = \sqrt{3}/\beta\left\{\frac{\pi}{2} + \arcsin\left[\left(1 + \left(\frac{\beta\epsilon}{a}\right)^2\right)^{-1/2}\right]\right\}, \tag{6}$$

and we can easily verify that $dI/d\omega < 0$. Thus, the solitons (4) and (5) are unstable. They apparently decay into periodic waves $\psi = \psi_0 e^{i\omega t - ikx}$ for which the Lighthill stability criterion is fulfilled:

$$\left(\frac{\partial\omega}{\partial\psi_0^2}\bigg/\frac{\partial^2\omega}{\partial k^2}\right)_{\psi_0\to 0} > 0.$$

2. Another approach to the problem under consideration is possible. As is known, in quantizing the nonlinear systems we arrive at a problem of one-dimensional gas of interacting bosons.[6,7] The ordinary solitons correspond to a bound, multiboson state in a system with a pair attraction between the particles. The three-particle repulsion and pair attraction were examined in Ref. 8. We shall discuss the opposite case of pair repulsion and three-particle attraction. Such a system can be described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + 2Va\sum_{i<j}\delta(x_i - x_j) - 3Wa^2\sum_{i<j<s}\delta(x_i - x_j)\delta(x_i - x_s), \tag{7}$$

where m is the mass of the particle, $V > 0$, $W > 0$, a is the interaction radius, and N is the number of particles. In the self-consistent field approximation in which all the single-particle wave functions are the same, the Hartree equation has the form

$$\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + \nu\phi - 2NV\phi^3 + \frac{3}{2}N^2W\phi^5 = 0, \tag{8}$$

where $\phi(x)$ is a normalized, single-particle, Hartree wave function ($\int\phi^2 dx = a$) and ν is the single-particle Hartree energy which is related to the total energy by the relation

$$E = N\nu - \frac{N^2V}{a}\int dx\,\phi^4 + \frac{N^3W}{a}\int dx\,\phi^6. \tag{9}$$

It can be seen that after we substitute $\omega = -1 - 2m\nu/\hbar^2$, $|\alpha| = 4mNV/\hbar^2$, and $\beta^2 = 3mN^2W/\hbar^2$ Eq. (8) coincides with Eq. (3) and has localized solutions such as Eqs. (4) and (5). Investigation of their stability reduces to a comparison of the energy per particle in such a bound state with the energy of a free boson. Substituting solutions (4) and (5) in expression (9) and using the normalization condition, we obtain

$$E/N = (V^2/2W)\left[1 + \tan\left(\sqrt{mW}\frac{aN}{\hbar}\right) \Big/ \left(\sqrt{mW}\frac{aN}{\hbar}\right)\right], \tag{10}$$

where [because of Eq. (6)] N can vary only in a limited range $N^*/2 < N < N^*$, $N^* = \pi\hbar/a(mW)^{1/2}$. The power soliton corresponds to $N = N^*$ and to the minimum energy per particle $(E/N)_{\min} = V^2/2W$. Since it is positive and enters the continuous spectrum of single-particle excitations, the quantum, exponential, and power solitons examined here are absolutely unstable.

3. In conclusion, we shall investigate the frequently examined[9] nonlinear wave equation for the complex amplitude by immediately selecting a nonlinear potential in it in the form that was discussed in Sec. 1

$$\ddot{\Psi} - \Psi'' + \Psi + a|\Psi|^2\Psi - \beta^2|\Psi|^4\Psi = 0. \tag{11}$$

Limiting ourselves to monochromatic solutions $\Psi = v(x)e^{i\Omega t}$, we obtain Eq. (3) and solutions (4) and (5) for $v(x)$ after substituting $\omega \to -\Omega^2$. Equation (11) describes the dynamics of the Lagrangian system, and we can easily determine the energy and the adiabatic invariant by using the corresponding Lagrangian (9). For our solutions they have the form

$$E = \int_{-\infty}^{\infty} dx \left[(v')^2 + (1+\Omega^2)v^2 + \frac{a}{2}v^4 - \frac{\beta^2}{3}v^6\right], \tag{12}$$

$$J = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_0^T dt \left[\dot{\Psi}\frac{\partial L}{\partial \dot{\Psi}} + \dot{\Psi}^*\frac{\partial L}{\partial \dot{\Psi}^*}\right] = 2|\Omega|\int_{-\infty}^{\infty} dx\, v^2, \tag{13}$$

where L is the density of the Lagrangian function.[9] In contrast to the classical and quantum analysis of solitons in the preceding sections, here we use the quasi-classical approach. As is known, in the quasi-classical approximation the quantization conditions reduce to the requirement $J = \hbar N$, where N is the number of quasi particles. Treating a soliton as a bound state of a large number of quasi particles, we calculate the energy in it per particle. Using Eq. (6), we find from Eqs. (12) and (13)

$$N = \frac{2|\Omega|}{\hbar}\frac{\sqrt{3}}{\beta}\left\{\frac{\pi}{2} + \arcsin\left[1\Big/\sqrt{1 + \frac{16}{3}(\beta/a^2)(1-\Omega^2)}\right]\right\}; \tag{14}$$

$$E/N = \frac{\hbar}{2|\Omega|}(1 + \Omega^2 + 3a^2/16\beta^2) + \frac{3a}{4\beta^2}\frac{\sqrt{1-\Omega^2}}{N} \tag{15}$$

N varies from 0 to $N_0 = 2(3)^{1/2}\pi/\hbar\beta$ which corresponds to a power soliton with a minimum energy per particle

$$(E/N)_{\min} = \hbar\left\{1 + \frac{3}{32}\left(\frac{\alpha}{\beta}\right)^2\right\}.$$

Since $(E/N)_{\min}$ is larger than the energy of the free particle $E_0 = \hbar$, the bound state is unstable with respect to the decay to free particles. (Notice that $dN/d\,|\Omega| > 0$ follows from Eq. (14)—as shown in Ref. 9, the solution in this case is unstable with respect to modulation.)

The results obtained by us for specific dynamic systems indicate that investigation of the stability of solitons in systems allowing power solitons is of crucial importance.

In conclusion, we express our gratitude to A. M. Kosevich for his attention and interest in this work.

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