Structure of steady-state solitons in systems with a nonlocal nonlinearity

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It is shown that steady-state, bound multi-soliton formations exist in physical systems that are described by the Schrödinger equation with a nonlocal nonlinearity.

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In recent years, an interest has increased considerably in the investigation of solitons-formations which play a fundamental role in the dynamics of many nonlinear processes (see, for example, the topical issue[1]). In this paper we examine the structure of the solitons under the conditions of nonlocal nonlinearity, i.e., when the system has a natural nonlinear scale. We show that the steady states exist in the form of a set of bound solitons, which is of interest, in particular, in studying plasma turbulence, construction of models for field particles,[1-3] etc.

Let us examine the Schrödinger equation as the original equation:

$$-i\Psi_t + \Delta\Psi - \phi\Psi = 0, \qquad (1)$$

in which the potential ϕ satisfies the relation

$$\Delta\phi - \phi = -\alpha\Delta|\Psi|^2 + \beta|\Psi|^2. \qquad (2)$$

Equations (1) and (2) are model equations for description of a number of effects in different physical systems. They can be obtained, for example, in the investigation of the self-stress of intensive Langmuir waves that propagate in a collisionless, magnetized plasma perpendicularly to the constant magnetic field. The parameter of nonlocality in this case is the Larmor radius of electrons. Of greatest interest is the case when $\alpha = 1$, $\beta \ll 1 (\beta = v_S^2/v_A^2, v_S$ and v_A are the acoustic and the Alfvén velocities), which specifically corresponds to the ionospheric plasma conditions, and is also fulfilled in experiments on plasma confinement and heating in thermonuclear devices.

At $\alpha = 0$ and $\beta = 1$ Eqs. (1) and (2) describe the thermal self-stress of a packet of electromagnetic waves in the plasma and in the condensed media (see, for example, Ref. 4); here the three-dimensional scale in the material binding (2) is the thermal conductivity length. These equations correspond to the Hartree model for nucleons with the wave function Ψ in a self-consistent field of nuclear forces with a Yukawa-type potential. Here the natural nonlinear scale is determined by the range of nuclear forces (Compton length of the pion wave).

Let us examine the one-dimensional, localized solutions of the original system[1] of the type $\Psi = \Phi(x)\exp(-iEt)$ and $\phi = \phi(x)$, which satisfy the equation

$$\Phi_{xx} - (E + \phi)\Phi = 0; \quad \phi_{xx} - \phi = -\alpha\Phi^2_{xx} + \beta\Phi^2; E > 0 \qquad (3)$$

and discuss in more detail the two limiting cases.

At $\alpha = 1$ and $\beta = 0$, we can easily determine from the first integral of Eq. (3) that only the even, localized distributions can be realized. A qualitative representation of the structure of such solutions can be obtained for sufficiently narrow (for the Larmor radius scale) field distributions in the upper hybrid soliton. It is convenient in this case to use the equation

$$\Phi_{xx} + (\Phi^2 - E - u)\,\Phi = 0, \quad u = \frac{1}{2}\int_{-\infty}^{\infty} \Phi^2(x')\exp(-|x - x'|)\,dx' \qquad (4)$$

which is equivalent to Eq. (3). It is clear that when $E \gg 1$ the integral part of the nonlinear term is a smooth [for the characteristic scale of the soliton $(\sqrt{E})^{-1}$] function. Therefore, the soliton solutions of Eq. (4) can be constructed in the adiabatic approximation, if we treat u as a slowly varying parameter. In the phase plane (Φ_x, Φ), which is determined by the local value of u, the solitons correspond to the integral curves that go out of the saddle point (0,0) and return to the original state after a specified number of revolutions inside a slowly varying separatrix. Thus, for one revolution we have a single soliton with a field distribution, which is asymptotically close ($E \to \infty$) to the distribution in the Langmuir soliton $\Phi = \sqrt{2E}\cosh^{-1}\sqrt{E}x$. Two revolutions correspond to a steady-state bisoliton, in which, because of nonlocal coupling, two ordinary solitons are captured in the density wells at a certain distance from each other, etc.

Figure 1 shows a localized solution (fourth mode) of Eq. (4), which was obtained by a numerical calculation on a computer. The amplitude of the solitons, which produce a multi-humped structure, decreases with decreasing E and more complex formations comprised of a set of N soliton objects appear (Fig. 2).[2] When $E \to 0$ the solitons expand and overlap each other to such an extent that the spatial distribution becomes

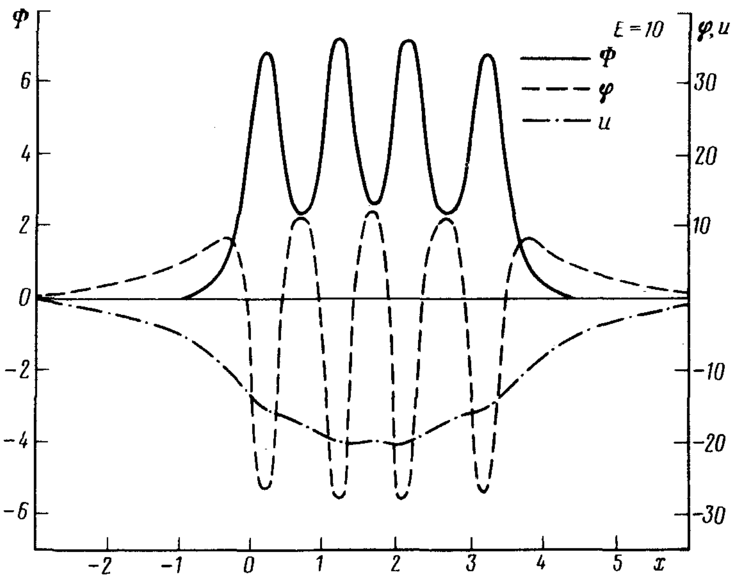

FIG. 1

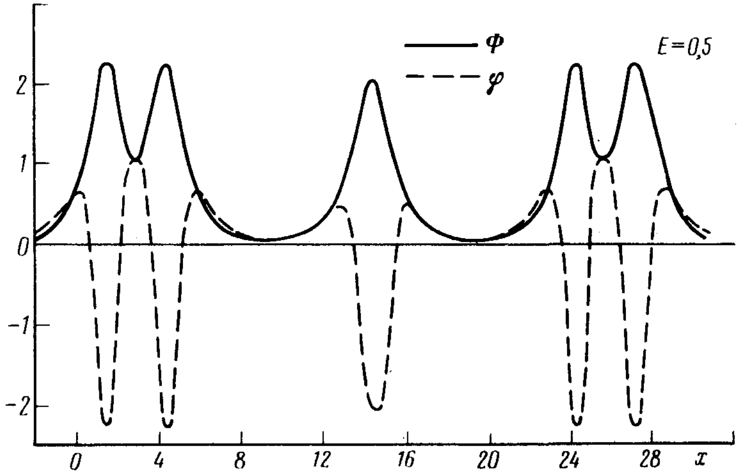

FIG. 2.

quasi homogeneous ($\Phi \to \Phi_0 = 1/\sqrt{2}$). Thus, for any value of E, i.e., for detuning of the frequency of the electric field from the plasma frequency, there is a denumerable set of different solitons of Langmuir waves that propagate with a subsonic velocity in a low-pressure magnetized plasma ($\beta \ll 1$) in the perpendicular direction to the magnetic field.

In the case of $\alpha = 0$ and $\beta = 1$ the transition to a local nonlinearity is realized when $E \to 0$, and in the entire range of parameters $0 < E \leqslant 1$ only one even soliton solution, which allows for $E = 1$ an exact analytic expression, is possible

$$\Phi = \phi = (3/2) \cosh^{-2}(x/2). \qquad (5)$$

All the eigenvalues $E > 1$ are infinitely degenerate. As the numerical calculations show, the eigenfunctions are "put together" by successive "addition" of a single soliton in such a way that the oscillation phases in the individual clusters of the field are shifted by π (the dot-dash curve in Fig. 3 represents the single soliton), i.e., there is almost a superposition of such formations (in the nonlinear system!). At $E > 1$ the distance between the solitons increases, so that in the limit an ordinary soliton with the distribution (5) remains.

The stability of the bound state can be easily investigated in the bisoliton solution for $E \approx 1$. Thus, the distance between the solitons $\bar{x} = 2\int_0^\infty x|\Psi|^2 dx$ is much greater than their size, and the interaction between them occurs via their "tails." Going over from the Schrödinger equation to the equation of motion of solitons of opposite polarity and different shape (5), we obtain

$$\frac{d^2\bar{x}}{dt^2} = \frac{81}{2} \exp(-\sqrt{E}\bar{x}) - \frac{81}{16} \exp(-\bar{x}). \qquad (6)$$

It can be seen that the stable, bound state is possible only when $E > 1$, and the distance between the solitons is $\bar{x}_0 = \ln 8/(\sqrt{E'} - 1)$, which coincides with the numerical calculations. Moreover, since these results are determined mainly by the asymptotic

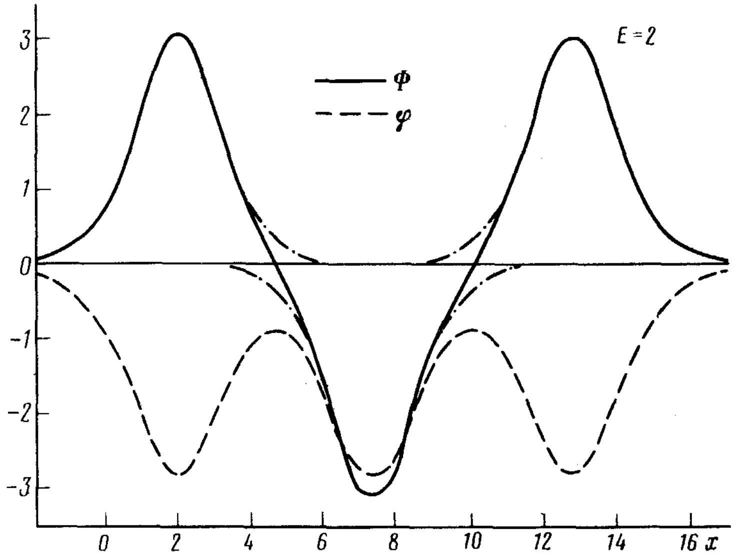

FIG. 3.

($x \to \pm \infty$) behavior of the solutions of Eq. (3), we can expect that they are qualitatively valid for the two-dimensional and three-dimensional solitons.

Thus, the nonlocality of nonlinear interaction in the Schrödinger equation produces a striking variety of localized solutions obtained by almost complete superposition of the single solitons (see Figs. 1, 2, and 3). We expect that this will prove to be important in understanding the new, nontrivial properties of interaction of the particle-like formations.

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[1)]The investigated solitons are characterized by an exponential asymptotic form ($x \to \pm \infty$) of the Φ and ϕ functions. The localized solutions in the fourth-order Hamiltonian systems, which have an asymptotic form of type $\Phi,\phi \sim \cos k_1 x \exp(-k_2 x)$ or $\Phi \sim \exp(-kx), \phi = A \cos kx$, were examined elsewhere.[5,6]

[2)]For brevity, we shall skip the discussion of this problem. A more complete investigation, including the modulation instability and generation and interaction of solitons, will be discussed in a comprehensive report.

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