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Spatial Localization of Nonlinear Waves in Layered and Modulated Media

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We describe analytically the nonlinear dynamics of the stationary waves propagating in an anharmonic medium along layered or modulated periodic waveguide system. The exact dynamical equations for discrete systems of coupled anharmonic oscillators, which are typically used for the description of real experiments on the localization of powerful optical fluxes, are analytically derived on the microscopic level and the conditions of their applicability are obtained. The criterion of the appearance of a spatially localized state in such systems is derived.

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In recent years, interest in researches on the theory of nonlinear waves and solitons is mainly concentrated on studying the nonlinear dynamics of real physical systems with their discreteness, defect character, internal microstructure, and spatial inhomogeneity. Of special interest are layered structures of different types promising for technological applications [1 – 6]. Examples of such media are multilayer magnets [1], which are used for the creation of elements for data storage and readout based on magnetooptical properties of such multilayer materials, layered crystals with polyatomic unit cells (in particular, high-T_c superconductors and their isostructural analogs, which contain layers with substantially different conducting and elastic properties [2]), corrugated optical fiber waveguides in nonlinear optics, etc. The simultaneous effect of the layered nature of the medium, which substantially alters the spectrum of its linear waves and their dispersion, and the nonlinearity of the medium can give rise to new physical effects such as dependence of the transparency of the medium on the power of the wave being transmitted [6], spatial localization of nonlinear waves (powerful light pulses) in periodic arrays of optical waveguides [4, 5] and the existence of so-called "gap solitons" [3]. The study of all these experimental works has given rise to the present investigation of such systems from the theoretical point of view.

In this Letter, we present the analytical analysis of the spatial localization of nonlinear stationary waves propagating in an anharmonic medium containing thin plane-parallel layers having different linear properties from the characteristics of the medium itself. Owing to the simultaneous appearance of linear localization at the defect layers (in the case when the layers play the role of waveguides) and nonlinear localization due to the anharmonicity of the medium around the layers, it becomes possible to have a resultant localization of the wave flux in a region containing a large number of plane layers. This effect has been observed experimentally in planar nonlinear optical waveguides with a periodically modulated cross section [4, 5]. The theoretical description of the nonlinear properties of layered structures is typically done using discrete models for the wave amplitudes in the individual waveguides [4, 5, 7] which are described phenomenologically by difference equations with arbitrary parameters. Under a number of simplifying assumptions a consistent derivation of these equations has been done in the simplest case when the nonlinearity is taken into account only in thin layers separated by wide regions of linear medium [6, 8]. We consider the case when all layered medium is substantially nonlinear and it is a nontrivial problem to find the effective nonlinearity of the individual waveguides and their effective interaction. The equations describing the propagation along the layered structure of a nonlinear monochromatic wave with an envelope that is slowly varying in space and time are derived both for (I) nonlinear optical medium containing plane-parallel waveguides, i.e., layers characterized by a larger refractive index than the optical medium between them, and for (II) optical waveguide of variable cross section. Material parameters of such layered structures are periodically modulated in

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the direction perpendicular to the direction of the wave propagation.

Consider the propagation of a nonlinear electromagnetic wave along the layered nonlinear optical medium containing plane-parallel waveguides, i.e., defect layers characterized by the refractive index differs from that one in the optical medium between them. We assume that the layers lie perpendicular to the z axis. In the case of a plane-polarized wave propagating in a nonmagnetic medium ($\mu = 1$) along the layers (in the x direction), with its electric field vector $\mathbf{E}$ directed along the y axis ($\mathbf{E}\|\mathbf{i}_y$), Maxwell's equations take the form

$$n^2(z,\mathbf{E}) \cdot \partial^2\mathbf{E}/\partial t^2 - c^2 \cdot \Delta\mathbf{E} = 0, \qquad (1)$$

where the refractive index n depends on the coordinate z and the electric field: $n = n_0 + n_1(z) + n_2(\mathbf{E})$, with $n_1(z) = n_1$ in the waveguides and $n_1(z) = 0$ outside them. (Defect layers represent optical waveguides for $n_1 > 0$.)

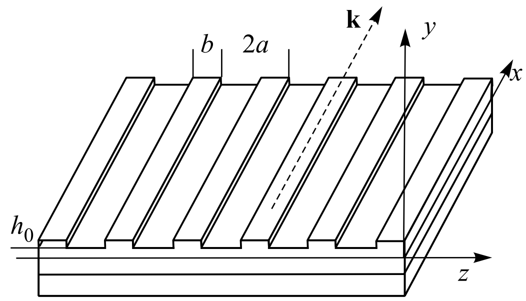
We assume that the modulation of the parameters of the medium and the energy density in the wave are small, i.e., $n_1, n_2 \ll n_0$, and the dependence of n on z needs to be taken into account only in the linear refractive index. We limit discussion to solutions in the form of nearly monochromatic waves with a fixed wave vector $\mathbf{k} = \mathbf{i}_x k$. In terms of the electric field E, which is slowly varying with z and t, the nonlinear contribution to the refractive index takes the form [9]: $n_2(E) = \sigma \cdot \alpha(\omega) \cdot |E|^2$, where $\sigma = \pm 1$ for focusing and defocusing media, correspondingly, and $\alpha(\omega)$ is the nonlinear permittivity. If the thickness b of the defect layers (optical waveguides) is much smaller than the distance $2a$ between them, then, introducing the new time $\tau = t \cdot \alpha ck/2n_0^2$ and the new coordinate $\xi = z \cdot k\sqrt{\alpha/n_0}$, we can reduce Eq. (1) to the following standard nonlinear Schrödinger equation

$$i\frac{\partial E}{\partial \tau} + \frac{\partial^2 E}{\partial \xi^2} + 2\sigma \cdot |E|^2 E = -\sum_n \lambda \cdot \delta(\xi - 2\tilde{a}n) \cdot E, \quad (2)$$

where parameters $\lambda = 2b(n_1/\sqrt{\alpha n_0})k$ and $\tilde{a} = a\sqrt{\alpha/n_0} \cdot k$.

In real optical experiments the statement of the problem may be somewhat different [4]: a nonlinear electromagnetic wave propagating in a plane optical waveguide of variable cross section (Figure). A nonlinear optical medium with refractive index $n = n_0 + n_2(\mathbf{E})$ occupies the region $0 < y < h(z) = h_0 + \Delta(z)$, where $\Delta > 0$, and the plane-polarized wave propagates along the x-axis. If the optical waveguide is bounded by an optically nontransparent medium, then, in the case of weak modulation of the layer thickness, solutions close to a monochromatic wave can be written in the form

$$\mathbf{E} = \mathbf{i}_y \cdot \{E_1(z,t) \cdot \cos(kx - \omega_* t) -$$



Optical waveguide of variable cross section

$$-E_2(z,t) \cdot \sin(kx - \omega_* t)\} \cdot \sin[\pi y/h(z)], \qquad (3)$$

where $E = E_1 + iE_2$, $E_{1,2}$ varies slowly with z and t, and we have chosen a relation $\omega_*(k) \approx (c/n_0) \cdot [(\pi/h_0)^2 + k^2]^{1/2}$. (Here the slow dependence $E_{1,2}(t)$ takes into account the difference of the true frequency $\omega(k)$ at a given k from the frequency $\omega_*(k)$ due to nonlinear effects and modulation of the parameters of the medium.)

Introducing the new coordinate $\xi = z \cdot (\omega_*/\mathrm{c}) \times \sqrt{2\alpha n_0/3}$ and time $\tau = t \cdot \alpha\omega_*/3n_0$, we can reduce Eq.(1) for the slowly varying function E, after its integration over the thickness of the waveguide, to the following standard equation

$$i\frac{\partial E}{\partial \tau} + \frac{\partial^2 E}{\partial \xi^2} + 2\sigma \cdot |E|^2 E = -\lambda(\xi) \cdot E, \qquad (4)$$

where, in the case of weak modulation of the function $h(z)$ ($\Delta_{\max} \ll h_0$), $\lambda(\xi) = [3\mathrm{c}^2/\alpha\omega_*^2 h_0^3] \times \Delta\{(\mathrm{c}/\omega_*)\sqrt{3/2\alpha n_0} \cdot \xi\}$. Consequently, the thicker regions of the optically transparent material play the role of effective waveguides in the two-dimensional nonlinear optical system under consideration, but the function $\lambda(z)$ can be replaced by a set of δ-functions only if these regions rather wide.

Thus for a slowly varying in space and time envelope of a nonlinear monochromatic wave propagating along the periodic layered structure (along the x-axis) containing identical plane-parallel layers differing in their linear properties from the surrounding matrix and separated by a distance much greater than their thickness, the evolution nonlinear Schrödinger equation (which is standard in soliton theory) is derived containing disturbances in the form of the spatial δ-functional array:

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial z^2} + 2\sigma|u|^2 u = -\sum_n \lambda \cdot \delta(z - 2an) \cdot u, \quad (5)$$

where the z-axis is directed perpendicular to the defect layers; the sign function $\sigma = \pm 1$ for “focusing” and “defocusing” media, respectively; the plane defect is characterized by $\lambda > 0$ in the case when the narrow layers have waveguide properties (they "attract" linear waves); $2a$ is the distance between the plane defect layers. For simplicity, in Eq. (5) we used the initial symbols of variables z and t instead of ξ and τ. Therefore, in order to get back to the initial parameters of the system one should use the transformations with whose help Eqs. (2) and (4) were obtained.

For waves of a stationary profile the problem is equivalent to the study of nonlinear excitations in a one-dimensional system containing point defects (nonlinear local oscillations). For a single isolated defect this problem has been investigated in Refs. [10] for arbitrary signs of σ and λ. In the case of periodic system of defects (defect layers) interacting through a nonlinear field, the solution of the problem becomes more complicated, and the problem is reduced to one of solving the nonlinear Schrödinger equation (5) for stationary localized states of the form $u(z,t) = u(z)\exp(-i\omega t)$ in the region outside the distinctive (defect) layers, with the following boundary conditions at them (at $z = 2an$):

$$\begin{gathered} u|_{2an-0} = u|_{2an+0}, \\ (\partial u/\partial z)|_{2an+0} - (\partial u/\partial z)|_{2an-0} = -\lambda u|_{2an}. \end{gathered} \qquad (6)$$

Let us consider the case of focusing ($\sigma = +1$) medium. Then the solution of Eq. (5) in the region n, where $2an \le z \le 2a(n+1)$, have the following form:

$$\begin{gathered} u_n(z,t) = u_n(z)\exp(-i\omega t) = \\ = \xi_n dn\{\xi_n(z - z_n),\, q_n\}\exp(-i\omega t). \end{gathered} \qquad (7)$$

Here $dn(p,q)$ is the Jacobi elliptic function with modulus q, parameter $\xi_n = \varepsilon/(2-q_n^2)^{1/2}$ characterizes the amplitude of the wave, and parameter ε is related to the value of ω: $\varepsilon = \sqrt{-\omega}$. The weak coupling of the waveguides (essential difference of the field in the waveguides and between them) means strong distinction of values of the elliptic function $dn(p,q)$ in different points, i.e., the closeness of its modulus to unity: $q'_n = \sqrt{1-q_n^2} \ll 1$, $\xi_n \approx \varepsilon$. Parameter z_n (which is different for various pairs of adjacent waveguides) characterizes the skewness of field distribution between these adjacent waveguides and the difference of the field amplitudes inside them.

A convenient characteristic of the localized wave is provided by the field amplitude $U_n = u(z = 2an)$ inside the n th waveguide which divides the regions with numbers $(n-1)$ and n. Then, using the boundary conditions (6) at the n th waveguide (taking into account the form of solution (7) and the definition of the field amplitude U_n), we can eliminate the modula q_n and link the pair of parameters (U_n, z_n) with the previous one, (U_{n-1}, z_{n-1}). It allows us to solve the problem, for example, numerically. On the other hand, in the limit of weak dynamic coupling between waveguides ($q'_n \ll 1$) and the condition of small-amplitude waves, $U_n \ll \varepsilon$, all the calculations are simplified and can be carried out analytically. Taking the solution (7) for u_n at the points $z = 2an$ and $z = 2a(n+1)$, we can easily express the parameter q_n in the region n through U_n and U_{n+1}:

$$q'_n = (2/\varepsilon)\sqrt{U_n U_{n+1}}\cdot\exp(-\varepsilon a). \qquad (8)$$

In a focusing medium, in which the frequency of the wave decreases as its amplitude grows, the condition $q'_n \ll 1$ actually implies the inequality $\varepsilon a \gg 1$ (the dynamical coupling of the waveguides decreases with increasing amplitude). As the further investigation is carried out just in this approximation, let us determine validity conditions of the inequality $\varepsilon a \gg 1$ in the real physical experiments. For this purpose we need to return to the transformations of the coordinate and of the parameter λ which have been used for obtaining Eq. (2). In the case of small-amplitude envelope solitons under consideration the inequality $\varepsilon a \gg 1$ reduces to the inequality $\lambda\tilde{a} \gg 1$. Returning to the initial variables of Eq. (2), we transform this inequality in the case of a system of optical waveguides to the following form:

$$2ab\cdot k^2\cdot 2n_1/n_0 = (8\pi^2 2ab/\mu^2)\cdot n_1/n_0 \gg 1, \qquad (9)$$

where $\mu = 2\pi/k$ is the wavelength of the propagating light.

For the systems of optical waveguides with $n_1 \sim n_0$ and $a \sim b$ for visible light from (9) the inequality follows: $a, b \gg 3\cdot 10^{-6}$ cm. This relation holds for the waveguide arrays of Refs. [4] having $a, b \sim 5\cdot 10^{-4}$ cm. If one compares formulas (2) and (4), then the relation between the changing of the waveguide thickness Δh and the corresponding effective changing of the refractive index in incrassate regions can be written in the form

$$\frac{n_1}{n_0} = \frac{\Delta h}{h_0^3 k^2} = \frac{1}{4\pi^2}\frac{\Delta h \mu^2}{h_0^3}. \qquad (10)$$

Substituting this relation into (9), we come to the following condition:

$$J = \frac{2ab}{h_0^2}\cdot\frac{2\Delta h}{h_0} \gg 1. \qquad (11)$$

For the waveguides of Refs. [4] having $b = 4\mu m$, $2a = (8 \div 11)\mu m$, $\Delta h \cong 1\mu m$ and $h_0 \cong 2\mu m$, we obtain $J = 8 \div 11 \gg 1$. Therefore, our approximation of weakly coupled waveguides holds for the description of the indicated experiments.

Using solutions (7) in the regions $(n-1)$ and n, we can rewrite the boundary conditions (6) as follows:

$$\sqrt{\xi_n^2 - U_n^2}\sqrt{U_n^2 - \xi_n^2 {q'}_n^2} +$$

$$+\sqrt{\xi_{n-1}^2 - U_n^2}\sqrt{U_n^2 - \xi_{n-1}^2 {q'}_{n-1}^2} = \lambda U_n. \qquad (12)$$

Substituting expressions for q'_n (8) and for $\xi_n = \varepsilon/\sqrt{1+{q'}_n^2}$ into Eq. (12), we obtain the equation for the amplitudes U_n and $U_{n\pm1}$, containing as parameters the frequency characteristic ε, waveguide parameter λ and the distance between waveguides $2a$. Allowing for the inequalities $U_n \ll \varepsilon$ and $\varepsilon a \gg 1$, this equation for the frequencies close to the lower edge of the continuous spectrum, $\omega_0 \approx -\lambda^2/4 - \lambda^2\exp(-\lambda a)$, can be reduced to the following one:

$$(\omega_0 - \omega)U_n - U_n^3 + v_0(2U_n - U_{n-1} - U_{n+1}) = 0. \quad (13)$$

Here $v_0 = (\lambda^2/2)\exp(-\lambda a)$ is a parameter characterizing the effective interaction of the waveguides via the nonlinear field. Since we are investigating only stationary states with a time dependence $\sim \exp(-i\omega t)$, the system of algebraic Eqs. (13) corresponds to the phenomenological system of dynamical equations

$$i\frac{\partial U_n}{\partial t} - \omega_0 U_n - v_0(2U_n - U_{n-1} - U_{n+1}) + |U_n|^2 U_n = 0. \quad (14)$$

Thus, in the case of weak dynamic coupling between defect planes the problem is reduced to an effective system for an infinite chain of coupled anharmonic oscillators (rotators).

The theoretical description of experimental results in [4, 7] was carried out in the framework of the discrete nonlinear system of coupled anharmonic oscillators described by the equation

$$i\frac{\partial U_n}{\partial t} + \beta U_n + C(U_{n-1} + U_{n+1}) + \gamma|U_n|^2 U_n = 0 \quad (15)$$

for the electric field U_n in the n th waveguide. This equation is of the same form as Eq. (14) obtained by us, but in works [4, 7] Eq. (15) was actually postulated and the phenomenological parameters contained in it were found by comparing with the experimental data.

The parameters of the system of different Eqs. (14), describing an infinite chain of coupled anharmonic oscillators, are all determined through the microscopic characteristics of layered medium and propagating wave (linear and nonlinear refractive indexes, transverse size of the waveguides and the distance between them, the wavelength of the propagating light). This allows us to compare the analytical results with the experimental data. The comparison of Eq. (14) derived by us from microscopic considerations with the phenomenological equation (15) results in the following dependences of the parameters of Eq. (15) on the material parameters of the system under consideration:

$$C/\gamma = v_0 = \lambda^2\exp(-\lambda a)/2 = \\ = [b^2 n_0(\Delta h)^2/2\alpha h_0^6 k^2]\cdot\exp(-4ab\Delta h/h_0^3), \qquad (16)$$

$$\beta/\gamma = -\omega_0 - 2v_0 = \lambda^2/4 = b^2 n_0(\Delta h)^2/4\alpha h_0^6 k^2.$$

Let us consider the case when the discrete Eqs. (13) can be reduced to the differential equation. This means that the localization region of the nonlinear wave in the layered medium is much larger than the period of this structure. This condition imposes an additional restriction on the wave amplitude: instead of the inequality $U_n \ll \varepsilon$ used by us under the derivation of Eq. (13), we now have a stronger inequality $U_n \ll \lambda\exp(-\lambda a/2)$. With the indicated stipulations, Eq. (14) can be replaced by the nonlinear Schrödinger differential equation for the function $U = U(Z,t)$ where the continuous coordinate $Z = 2an$ substitutes the discrete number n:

$$-i\frac{\partial U}{\partial t} - 4a^2 v_0\frac{\partial^2 U}{\partial Z^2} + \omega_0 U - |U|^2 U = 0. \qquad (17)$$

This equation has the well-known soliton solution of the form

$$U_n \approx \sqrt{2(\omega_0-\omega)}\cdot \mathrm{ch}^{-1}(n\sqrt{(\omega_0-\omega)/v_0})\cdot\exp(-i\omega t),$$

which describes analytically a nonlinear wave localized in the transverse direction and propagating along a layered structure. Such a bright spatial optical soliton ("supersoliton") was observed, in particular, in the experiments of Refs. [4, 7].

It is follows from (8) that in the case of weak "superlocalization" ($U_n \approx U_{n+1}$) we have $q' \approx \approx (4U/\lambda)\exp(-\lambda a/2)$. On the other hand, the parameter q' defines the minimum value of the elliptic function $dn(p,q)$. Therefore, the minimum field $u(z)$ between waveguides is equal to $u_{\min} = \xi q' \approx \lambda q'/2$. Then the ratio of the field inside of the waveguides to the field between them is of the order of $u_{\max}/u_{\min} \approx \exp(\lambda a/2)/2 \gg 1$. The state localized in a large number of waveguides represents, in fact, a "supersoliton", i.e., spatially coupled system of a few solitons each of them localized in the waveguide.

We particularly note that the spatial localization of the wave flux, in principle, is possible for an indefinitely small power of this flux – it is required only a sufficient closeness of the frequency ω to ω_0. However, such a localized excitation arises not for any initial characteristics of a flux (in our case – for power and profile of the injected light beam). The initial profile of a beam is essential in this case. For Eq. (17) the condition of the appearance of localized state reduces to the inequality

$$\int_{-\infty}^{+\infty} dZ|U| > 2\sqrt{2}a\sqrt{v_0}\ln(2+\sqrt{3}). \qquad (18)$$

Since in our designations $Z = 2an$ and the initial beam is localized in the experiments in the region of the order of the waveguide size, then the indicated inequality can be rewritten as follows: $u_{\max} > \lambda \cdot \exp(-\lambda a/2) \cdot \ln(2+\sqrt{3})$. Such a threshold appearance of the localized state is really observed experimentally. Unfortunately, the experimental data in [4] are cited only for several powers of the flux higher and lower of the critical value, but this value itself is not indicated.

In conclusion, we studied analytically the character of localization of nonlinear stationary waves propagating along a periodic system of thin plane-parallel defect layers (waveguides). It was shown that for weak coupling between waveguides the problem reduces to an effective periodic system for an infinite chain of coupled anharmonic oscillators. The system of difference equations was analytically derived for the description of such an oscillator chain. It was found the dependences of all the coefficients of these equations on the microscopic characteristics of layered medium and propagating wave. The spatial localization ("superlocalization") of the wave flux in such systems was investigated and the solution describing the wave localized in the transverse direction to its propagation was derived. The criterion of the appearance of localized state was obtained. Our results can be applied for the description of experiments on the localization of powerful optical fluxes in layered or modulated structures and in periodic arrays of optical waveguides.

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