Square roots of Hamilton's equations for supersymmetric systems

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The existence of dynamic equations is proved for supersymmetric Hamiltonian systems. These equations lead to Hamilton's equations of motion and they can be treated as a square root of these equations. © *1994 American Institute of Physics.*

1. The purpose of this note is to show that Hamilton's equations of motion, which describe the dynamics of a supersymmetric system, result from other dynamic equations of motion which are formulated in terms of the even Poisson–Martin bracket, but with the help of the supercharge Q^α, taken as a "Hamiltonian," and its corresponding Grassmann-odd derivative D^α is a square root of the time derivative. Since these latter dynamic equations of motion themselves are not a consequence of the Hamilton equations, they apparently represent a more restrictive level of the description for the dynamics of Hamilton's supersymmetric systems and can be treated as a square root of the Hamilton equations. This formulation of the dynamics shows that the supercharges have not only a symmetry meaning but also dynamic meaning.

2. Let us assume that the Hamilton supersymmetric system has a phase superspace with the real coordinates $z^M=(q^a,p_a;\theta^\alpha)$, where q^a,p_a $(a=1,\ldots,n)$ are even and θ^α $(\alpha=1,...,m)$ are odd (with respect to the Grassmann grating) canonical variables. Hamilton's equations of motion for this system are usually written on the basis of the even Poisson—Martin bracket

$$\{A,B\}=A\left[\sum_{a=1}^{n}(\overleftarrow{\partial}_{q^a},\overrightarrow{\partial}_{p_a}-\overleftarrow{\partial}_{p_a}\overrightarrow{\partial}_{q^a})-i\sum_{\alpha=1}^{m}\overleftarrow{\partial}_{\theta^\alpha}\overrightarrow{\partial}_{\theta^\alpha}\right]B, \tag{1}$$

using the even Hamiltonian $H(z)$ and give the evolution with the time t of an arbitrary quantity f, which depends on the variables z^M in the form

$$\frac{df}{dt}=\{f,H\}. \tag{2}$$

In definition (1) $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are right and left derivatives, and the notation $\partial_z=\partial/\partial z$ is introduced.

It immediately follows from (2) that the time is canonically conjugate with the Hamiltonian $H(z)$ in the bracket (1)

$$\{t(z),H(z)\}=1. \tag{3}$$

Since for every particular system the Hamiltonian $H(z)$ is the definite function of the phase coordinates z^M, the time t, in accordance with (3), must also be a definite function of z^M in the arbitrary function $t_0(H)$, which depends on the choice of the time origin.

In the transition from the coordinates $z^M=(q^a,p_a;\theta^\alpha)$ to real, canonical coordinates $z^{M'}(z)$, in the bracket (1), which contain among the canonically conjugate pairs a pair consisting of the time $t(z)$ and the Hamiltonian $H(z)$, it also follows from (2) that the rest of the canonical quantities $z^{M'}$ would be the integrals of motion for the system being considered: the even $I_1(z),\dots, I_{2(n-1)}(z)$ and the odd $\Theta^1(z),\dots,\Theta^m(z)$. To avoid a misunderstanding, we stress that the functions $z^{M'}(z)$ are considered as elements of the Grassmann algebra with the generators θ^α along the ring of the Grassmann-even functions that depend on the even variables q^a and p_a. Note that the general mathematical scheme allows us to introduce the terms with the odd functions of q^a and p_a into the θ^α expansions for $z^{M'}(z)$. However, we do not consider these terms because they necessarily include the constant Grassmann values whose physical interpretation encounters many difficulties. In terms of the new coordinates $z^{M'}(z)$, we can rewrite the bracket (1) as follows:

$$\{A,B\}=A\left[\overleftarrow{\partial}_t\overrightarrow{\partial}_H-\overleftarrow{\partial}_H\overrightarrow{\partial}_t+\sum_{k=1}^{n-1}(\overleftarrow{\partial}_{I_{2k-1}}\overrightarrow{\partial}_{I_{2k}}-\overleftarrow{\partial}_{I_{2k}}\overrightarrow{\partial}_{I_{2k-1}})+i\sum_{\alpha=1}^{m}\overleftarrow{\partial}_{\Theta^\alpha}\overrightarrow{\partial}_{\Theta^\alpha}\right]B. \tag{1a}$$

Under the condition of the problem the Hamilton system under consideration possesses supersymmetry whose generators $Q^\alpha(z)$ satisfy the superalgebra

$$\{Q^\alpha,Q^\beta\}=2i\delta^{\alpha\beta}H, \tag{4a}$$

$$\{Q^\alpha,H\}=0. \tag{4b}$$

A comparison with expression (1a) for the bracket reveals the relation between the supercharges Q^α and the canonical odd integral of motion Θ^α,

$$Q^\alpha=\sqrt{2H}\Theta^\alpha. \tag{5}$$

The infinitesimal canonical transformation of $z^{M'}$ with the generating function $\lambda(z^{M'})$ can be written with the help of the bracket in the form

$$\delta z^{M'}=\{z^{M'},\lambda\}.$$

Taking into account relation (5) and the expression for the bracket (1a), we can write the canonical transformation oft he supercharges Q^α in the form

$$\delta Q^\alpha=\{Q^\alpha,\lambda\}=iD^\alpha\lambda,$$

where the odd derivatives

$$D^\alpha=\partial_{\eta^\alpha}+i\eta^\alpha\partial_t$$

are the square roots of the derivative with respect to time t

$$i\partial_t=(D^\alpha)^2, \tag{6}$$

and the quantities $\eta^{\alpha}=Q^{\alpha}/2H$ contained in the definition of D^{α} have the meaning of the superpartners for t.

Using the superalgebra (4), the Jacobi identifies for the bracket (1), and the property of the odd derivative (6) we can rewrite the Hamilton equation of motion for the phase superspace coordinates z^M

$$\frac{dz^M}{dt}=\{z^M,H\} \tag{2a}$$

in the form

$$(D^{\alpha})^2 z^M=\{\{z^M,Q^{\alpha}\},Q^{\alpha}\}. \tag{7}$$

Note that here is no summation over the index α in relations (6) and (7).

If we postulate with the help of any supercharge Q^{α} and its corresponding odd derivative D^{α} the equations of motion for the system in the following way

$$iD^{\alpha}z^M=\{Q^{\alpha},z^M\}, \tag{8}$$

where the bracket is assumed to be expressed in terms of the variables $z^M=(q^a,p_a;\theta^{\alpha})$ according to (1), then the action of the odd derivative D^{α} on Eq. (8) and using (6), (8), and (4), we obtain the initial Hamilton equations (2a). Thus, Hamilton's equations (2a) follow from Eqs. (8), while the inverse statement, in general, is inapplicable.

3. As an illustration of the general consideration, we give a particular example of Witten's supersymmetric mechanics[1] in its classical version,[2] whose phase superspace consists of four real coordinates $z^M=(q,p;\theta^1,\theta^2)$. In the given case, there are no even canonical integrals of motion I, while the Hamiltonian H, the time , the supercharges Q^{α}, and the fermionic charge F can be written in terms of the coordinates z^M as follows:

$$\begin{aligned}
H&=\frac{p^2+W^2(q)}{2}+i\theta^1\theta^2W'(q),\\
t&=\int_{q_0}^{q}[2H(q,p;\theta^1,\theta^2)-W^2(q')]^{-1/2}dq'+\frac{i\theta^1\theta^2W(q)}{2Hp}+t_0(H),\\
Q^1&=p\theta^1+W(q)\theta^2,\quad Q^2=p\theta^2-W(q)\theta^1,\quad F=i\theta^1\theta^2.
\end{aligned} \tag{9}$$

Hamilton's equations (2a) for every component of z^M have the form

$$\dot{q}=p,\quad \dot{p}=-WW'-iW''\theta^1\theta^2,\quad \theta^1=-W'\theta^2,\quad \theta^2=W'\theta^1, \tag{10}$$

where the dot and the prime mean the derivatives with respect to t and q, respectively. The expressions for the coordinates z^M in terms of t, H, and $\eta^{\alpha}=Q^{\alpha}/2H$, obtained by inverting relations (9), are

$$\begin{aligned}
q&=q_1(t,H)+i\eta^1\eta^2q_2(t,H),\\
p&=p_1(t,H)+i\eta^1\eta^2p_2(t,H),\\
\theta^{\alpha}&=f^{\alpha}_{\beta}(t,H)\eta^{\beta}.
\end{aligned} \tag{11}$$

Here $q_{1,2}$, $p_{1,2}$ and f^α_β are the Grassmann-even real functions of t and H. Hamilton's equations (10) in terms of the coefficients in the η^α expansions (11) for z^M take the form

$$\dot{q}_1=p_1, \quad \dot{q}_2=p_2, \quad \dot{p}_1=-W(q_1)W'(q_1),$$
$$\dot{p}_2=\{q_2[W(q_1)W''(q_1)+(W'(q_1))^2]+W''(q_1)(f^1_1f^2_2-f^1_2f^2_1)\}, \tag{10a}$$
$$\dot{f}^1_1=-W'(q_1)f^2_1, \quad \dot{f}^1_2=-W'(q_1)f^2_2, \quad \dot{f}^2_1=W'(q_1)f^1_1, \quad \dot{f}^2_2=W'(q_1)f^1_2.$$

Let us consider the system of equations (8) corresponding to the supercharge Q^1

$$iD^1z^M=\{Q^1,z^M\}. \tag{8a}$$

In terms of the coefficients in the η^α expansions (11) for z^M, this system of equations is

$$W(q_1)=f^2_1, \quad q_2=f^1_2, \quad p_1=f^1_1, \quad p_2=-f^2_2W'(q_1),$$
$$\dot{q}_1=f^1_1, \quad q_2W'(q_1)=\dot{f}^2_2, \quad \dot{p}_1=f^2_1W'(q_1), \quad p_2=\dot{f}^1_2. \tag{12}$$

It can be easily seen that Hamilton's equations (10a) follow from equations (12), while the latter cannot be derived from (10a). Note also that the set of equations (8), which correspond to different supercharges Q^1 and Q^2, are not consequences of each other in the sense of differential equations. However, the algebraic connection between them can be established with the help of the fermionic charge F, which transforms the supercharges Q^1 and Q^2 into each other

$$\{F,Q^\alpha\}=\epsilon^{\alpha\beta}Q^\beta, \tag{4c}$$

$$\{F,H\}=0, \tag{4d}$$

($\epsilon^{\alpha\beta}=-\epsilon^{\beta\alpha}$, $\epsilon^{21}=1$), by using the Jacobi identities for the superalgebra (4).

4. In summary, we have shown that the dynamics of the supersymmetric Hamilton system can be described not only by the Hamilton equations (2a), but also by means of Eqs. (8), which can be treated as square roots of (2a). Note that the interrelation between Eqs. (2a) and (8) is analogous to that between the Yang–Mills equations and the self-duality (or anti-self-duality) equations. Solutions of Eqs. (8) therefore resemble the self-dual (or anti-self-dual) solutions of the Yang–Mills equations.

Note the following point. Since Hamilton's dynamics for the supersymmetric systems, with an equal number of pairs of even and odd phase coordinates, can equivalently be described on the basis of even and odd Poisson brackets,[2,3] the square roots of Hamilton's equations for these systems can apparently be obtained on the basis of the odd Poisson bracket, using the same odd derivatives D^α. Taking into account he duality between the even and the odd integral of motion for such systems upon changing the bracket parity,[2,3] we must assume in this case that the even supercharges $\bar{Q}^\alpha$, which are the linear combinations of the even Hamiltonian and other even integrals of motion, are the "Hamiltonians" in the dynamical square-root equations

$$iD^\alpha z^M=\{\bar{Q}^\alpha,z^M\}_1,$$

where $\{..., ...\}_1$ is the odd Poisson bracket.

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