Relaxation oscillations of solitons

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The relaxation of a perturbed (amplified) optical soliton is analyzed. A nonlinear interference of the soliton and a radiation field leads to oscillations of the soliton amplitude which are damped in a power-law fashion. A new effect is observed: a mutual attraction of solitons as a result of their interaction with a nonsoliton part. © *1994 American Institute of Physics.*

1. In several types of long, fiber-optic communications links currently under development, the bits of information are solitons. Optical solitons can be described well by the nonlinear Schrödinger equation[1,2]

$$iE_z+E_{tt}+2|E|^2E=0. \tag{1}$$

Equation (1) is written in dimensionless variables in a coordinate system moving at the group velocity. Compensation for linear damping is provided by amplifiers (Refs. 3 and 4, for example), whose effect is essentially a simple multiplication of the input signal by a gain coefficient γ. For example, the shape of an isolated soliton,[5]

$$E(t,z)=\frac{E_0 e^{iE_0^2 z}}{\cosh E_0 t}, \tag{2}$$

can be assumed to be unaffected by passage through an amplifier ($z=0$), while the amplitude of the soliton becomes

$$E_1(z=0,t)=\frac{\gamma E_0}{\cosh E_0 t}. \tag{3}$$

Here γ lies in the interval $1\geqslant\gamma\geqslant 1.5$. At large values of γ, new solitons are created.[6] If damping is ignored, then the behavior of the electromagnetic field after the amplifier is governed by Eq. (1) under initial condition (3). The amplifier disrupts the relationship between the width and the amplitude of the soliton. As a result, the soliton relaxes to new, matched values of the width and amplitude:

$$\tilde{E}_0=E_0(2\gamma-1). \tag{4}$$

As was pointed out in Ref. 6, the relaxation to the new state is oscillatory; the amplitude of the oscillations falls off slowly at large values of z.

In the present letter we examine these oscillations and the nonlinear interaction of the relaxing solitons. As we will show below, the basic reason for the relaxation oscillations is a nonlinear interference of a soliton with the radiation field. The relaxation process is fundamentally nonadiabatic, and the frequency of the oscillations differs from

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that found by variational methods.[7] The most important effect from our standpoint is the decrease in the distance between solitons due to their interaction with the radiation field, which is not exponentially small.

2. The solution of the Cauchy problem for Eq. (1) reduces[5] to the solution of the direct and inverse scattering problems for a linear operator:

$$\frac{\partial \psi}{\partial t} = i(\lambda \sigma_3 + \hat{E})\psi, \tag{5}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} 0 & E^* \\ E & 0 \end{pmatrix}.$$

Initial condition (4) differs from soliton solution (2), which is a nonreflecting potential $\hat{E}$, in that it contains a continuous-spectrum component in addition to points of a discrete spectrum, $\lambda_0 = iE_0(\gamma - 1/2)$. The scattering-matrix coefficients $a(\lambda)$ and $b(\lambda)$ for spectral problem (5) are[6]

$$a(\lambda) = \frac{\lambda - iE_0/2 - i\gamma E_0}{\lambda + E_0/2 + i\gamma E_0}\,\hat{a}(\lambda), \quad b(\lambda) = i \sin \pi\gamma/\cosh\left(\frac{\pi\lambda}{E_0}\right), \tag{6}$$

where the function

$$\hat{a}(\lambda) = \frac{[\Gamma(-i\lambda/E_0 + 1/2)]^2}{\Gamma(-i\lambda/E_0 + \gamma - 1/2)\Gamma(-i\lambda/E_0 - \gamma + 3/2)}$$

is analytic and has no zeros in the upper λ half-plane.

In the absence of a soliton, the nonsoliton part asymptotically (as $z \to \infty$) spreads out in a dispersive fashion:[8]

$$E_1(z,t) = \frac{1}{z^{1/2}} f(\xi) \exp\left\{ i\frac{t^2}{4z} + i\alpha(\xi)\ln z \right\} + O(z^{-1/2}), \tag{7}$$

where

$$\alpha(\xi) = 2|f(\xi)|^2, \quad \xi = -\frac{t}{4z},$$

and $|f(\xi)|$ is related to the coefficient $b(\lambda)$ by

$$|f(\xi)|^2 = -\frac{1}{4\pi}\ln[1 - |b(\xi)|^2]. \tag{8}$$

The asymptotic wave function corresponding to (7), which is analytic in the upper half-plane of the spectral parameter λ ($\mathrm{Im}\lambda > 0$), is, according to Refs. 9 and 11,

$$\psi = \Phi \exp[i(\lambda t + 2\lambda^2 z)\sigma_3], \tag{9}$$

where Φ is a 2×2 matrix with the components

$$\Phi_{11}=\exp\left[-i\int_{\xi}^{\infty}\frac{\alpha(\xi)}{\lambda-\xi}\,d\xi\right], \tag{10}$$

$$\Phi_{21}=\frac{E_1}{2(\lambda-\xi)}\exp\left[-i\int_{\xi}^{\infty}\frac{\alpha(\xi)}{\lambda-\xi}\,d\xi\right], \tag{11}$$

$$\Phi_{12}=\frac{-E_1^*}{2(\lambda-\xi)}\exp\left[-i\int_{-\infty}^{\xi}\frac{\alpha(\xi)}{\lambda-\xi}\,d\xi\right], \tag{12}$$

$$\Phi_{22}=\exp\left[-i\int_{-\infty}^{\xi}\frac{\alpha(\xi)}{\lambda-\xi}\,d\xi\right]. \tag{13}$$

Equations (9)–(13) are valid far from the resonance $\lambda=\xi$:

$$|\lambda-\xi|\gg|E(t,z)|\sim E_0 z^{-1/2}.$$

The addition of a discrete spectrum at the point $\lambda=\lambda_0$ with a given continuous spectrum reduces to the replacement[12,13]

$$\Phi\rightarrow\tilde{\Phi}=\chi\Phi,$$

where $\chi=1-\dfrac{\lambda_0-\bar{\lambda}_0}{\lambda-\lambda_0}P$. Here the projection operator P $(P^2=P)$ is defined by

$$P=\frac{|n\rangle\langle n|}{\langle n|n\rangle}, \tag{14}$$

and the vector $|n\rangle$ is specified by means of a constant complex vector $|n_0\rangle$:

$$|n\rangle=\psi(z,t,\lambda_0)|n_0\rangle.$$

The field $E(z,t)$ is expressed in terms of the component P_{21}:

$$E(z,t)=E_1(z,t)-2(\lambda_0-\bar{\lambda}_0)P_{21}. \tag{15}$$

Assuming a vector

$$|n_0\rangle=\begin{pmatrix} e^{i\gamma} \\ e^{-i\gamma}\end{pmatrix} \text{ and } \lambda_0=\zeta+i\eta,$$

and using (9)–(13), we find from (15)

$$E(z,t)=E_1(z,t)-2i\eta e^{-i\Phi_2}\left(e^{i\varphi}+\frac{E_1(z,t)}{2}\left[\frac{e^{\theta}}{\lambda_0-\xi}-\frac{e^{-\theta}}{\bar{\lambda}_0-\xi}\right]\right)$$
$$\times\left(\cosh(\theta-2\eta\Delta)-\frac{|E_1(z,t)|\eta}{|\lambda_0-\xi|^2}\sin(\Phi_1+\Phi_2-\varphi)\right)^{-1}. \tag{16}$$

Here

$$\theta=2\eta(t+4\zeta z)+2\gamma'',$$

$$\varphi=-2\zeta t+4(\eta^2-\zeta^2)-2\gamma',$$

$$\Phi_1 = \arg E_1,$$

$$\Phi_2 = \int_{-\infty}^{\infty} \mathrm{sign}(\xi' - \xi) \frac{(\xi' - \zeta)\alpha(\xi')d\xi'}{|\xi' - \lambda_0|^2},$$

$$\Delta(\xi) = -\frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(\xi' - \xi) \frac{\alpha(\xi')d\xi'}{|\xi' - \lambda_0|^2}.$$

The asymptotic solution in (16) refines the result of Ref. 10, which was from an analysis of conservation laws. It is important to note that the factors $\Phi_2(\xi)$ and $\Delta(\xi)$ cannot be found by this approach. Although corresponding calculations were carried out in Ref. 11, some terms on the same order of magnitude $(\sim z^{-1/2})$ were discarded from the result. As we will see below, those terms have an important physical meaning.

3. For initial conditions (3), which are symmetric with respect to t, the solution must be even with respect to t. We thus have $\gamma'' = \zeta = 0$. The quantity $E_1(z,t)$ and the phase $\Phi_2(\xi)$ are even functions of t, while $\Delta(\xi)$ is odd. The function $\alpha(\xi)$ in (16) is determined by (6) and (8), and we have $\eta = E_0(\Gamma - 1/2)$.

Solution (16) is the asymptotic form of the Cauchy problem at large z, determined within a constant γ' and a constant overall phase [which has been omitted from (16)]. *This solution is the result of a nonlinear superposition of a soliton and radiation.* The interaction between these two components leads to oscillations of the soliton amplitude,[1] which relax to the equilibrium value $\tilde{E}_0 = E_0(2\Gamma - 1)$. The oscillation frequency $\Omega = \tilde{E}_0^2 - \alpha(0)/z$ asymptotically approaches the soliton rotation frequency, while the oscillation amplitude falls off as $z^{-1/2}$. This interference is a very nonlinear effect: The amplitude of the solution reaches a maximum when the soliton and the radiation background have opposite phases.

4. We consider two relaxing solitons $(\zeta_{1,2} = 0)$ separated by a distance T $(TE_0 \gg 1)$. Clearly, the interaction due to the overlap of the solitons is exponentially small. The main effect arises from the interaction of the soliton and the nonsoliton part radiated by the other soliton (in this case, the interaction of the waves radiated by the relaxing solitons can be ignored; it arises in the next order of an asymptotic expansion in $z^{-1/2}$). As a result, the solitons move toward each other: The phases change. The shifts of the centers of the solitons can be found simply by using solution (16) after we set $\gamma'' = -\eta T$ in it. Equating the argument of the hyperbolic cosine to zero, we find the following equation for the coordinate of the center of the soliton, $t(z)$:

$$t(z) = T - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}\left(\xi + \frac{t(z)}{4z}\right) \frac{\alpha(\xi)d\xi}{\xi^2 + \eta^2}. \tag{17}$$

Differentiating this relation with respect to z, we find an expression for the "velocity" of the center of the soliton at an accuracy sufficient for our purposes:

$$v=\frac{dt}{dz}=-\frac{t}{4z^2}\frac{\alpha\left(-\frac{t}{4z}\right)}{\left(\frac{t}{4z}\right)^2+\eta^2}. \tag{18}$$

If the radiation is to the left of the soliton ($T>0$), the soliton velocity v is negative: The two relaxing solitons move toward each other.

In the case $\eta T \gg 1$, we can approximate t on the right side of (19) by T. As a result, the total shift of the soliton can be found through a simple integration:

$$\Delta t=-\int_0^\infty \frac{\alpha(\xi)}{\xi^2+\eta^2}\,d\xi. \tag{19}$$

For initial condition (3), the function $\alpha(\xi)$ is even. In this case the integral can be evaluated explicitly:

$$\Delta t=\frac{1}{2\eta}\log|\hat{a}(i\eta)|=\frac{1}{E_0(2\gamma-1)}\log\left(\frac{\Gamma^2(\gamma)}{\Gamma(2\gamma-1)}\right). \tag{20}$$

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[1)]A similar interference occurs for a soliton against a constant background[14] $E_0=A_0\exp(-2i|A_0|^2z)$.

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