Exact values of the conductivity exponents in the 2D case

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A relation is derived for the critical conductivity exponents t and q on the basis of a scaling hypothesis for critical fluctuations in the theory of second-order phase transitions and also on the basis of an analogy between percolation-theory problems and second-order phase transitions. This relation is $t+q=\nu d-\beta$. In the 2D case, this relation yields the exact value $q_2=t_2=91/72$. A violation of this relation for critical percolation of dimensionality $d=6$ is discussed. © *1994 American Institute of Physics.*

The effective conductivity of randomly inhomogeneous two-phase media near the percolation threshold behaves in a critical fashion:[1,2]

$$\sigma_e \approx \sigma_1 \tau^t \text{ for } \tau>0, \quad \sigma_e \approx \sigma_2 |\tau|^{-q} \text{ for } \tau<0, \tag{1}$$

where σ_1 and σ_2 are the conductivities of the phases $(\sigma_2/\sigma_1 \ll 1)$, $\tau=(p-p_c)/p_c$ is the distance from the percolation threshold p_c, and t and q are critical conductivity exponents—universal constants which depend on only the dimensionality of the problem.

At one time, a major effort was undertaken to relate the "dynamic" exponents t and q with the "geometric" ones, which characterize the divergence of the correlation length near $p_c-\nu$, the mean number of sites in a finite cluster $(S-\gamma)$, etc. For example, Stauffer[3] (see also Ref. 4) has collected expressions derived by various investigators for the critical exponents t and q. For t, these expressions are $1+\beta$, $(d-1)\nu$, $1+(d-2)\nu$, $1+2\beta$, $(5d-6\nu)/4$, and $[(3d-4)\nu-\beta]/2$. For q they are $2\nu-\beta$ and $\nu-\beta/2$. None of these expressions is satisfactory. It is shown below that considerations based on a scaling hypothesis lead to a relationship between the geometric exponents and the sum $t+q$. In the 2D case, this relation yields the exact value $t_z=q_2$.

This derivation is based on (a) assumptions regarding the validity of a scaling hypothesis for critical fluctuations in the theory of second-order phase transitions[5,6] and (b) on an analogy between percolation-theory problems and the problem of a second-order phase transition.[1,2,7]

According to (a) (see, for example, Ref. 6), the magnetization m near the point of the phase transition, T_c, is given by

$$m=\xi^{-d/\sigma} w_{\pm}(h\xi^{-d_h}), \tag{2}$$

where $\xi \sim \tau^{-\nu}$ is a correlation length, $\tau \sim T-T_c$, h is the external field (a magnetic field in the case at hand), $w_{\pm}$ are functions of dimensionality zero (Ref. 6, Ch. II, §3), and d_σ and d_h are critical exponents:

$$d_\sigma=(d-2+\eta)/2, \quad d_h=(d+2-\eta)/2. \tag{3}$$

In a region in which the phase transition is smeared, which exists only in the case $h\neq 0$, the argument of the functions $w_\pm$ reaches a value on the order of one. We denote the value of τ at which this event occurs by Δ, which is called the "smearing region" in percolation theory. In the smearing region (this is a smearing of the phase transition) we thus have $|\tau|\approx\Delta$; using $\xi\sim|\tau|^{-\nu}$, we find

$$\Delta\sim h^{1/\nu d_h}. \tag{4}$$

Hence the correlation length in this region is

$$\xi_c=\xi(|\tau|\approx\Delta)\sim h^{-1/d_h}. \tag{5}$$

According to Ref. 5, on the other hand, the correlation length in the smearing region is

$$\xi_c\sim h^{-\mu}, \quad \mu=2/(d+2-\eta). \tag{6}$$

We find, as we should,

$$\mu=\frac{2}{d+2-\eta}=\frac{1}{d_h}. \tag{7}$$

A corresponding situation (the equality $\mu=1/d_h$) should prevail in percolation theory. Here the ratio of the conductivities of the phases, $h=\sigma_2/\sigma_1\ll 1$, plays the role of a magnetic field.[1,2] In the limit $\tau\to 0$, but with $h\neq 0$, the quantities $\sigma_e(p<p_c)$ and $\rho_e(p>p_c)\equiv 1/\sigma_e(p>p_c)$ no longer go off toward infinity. In the smearing region, $\sigma_e(|\tau|\approx\Delta)$ has a finite value.

An analog of scaling function (2) for σ_e is[1,2]

$$\sigma_e\approx(\sigma_1^q\sigma_2^t)^{1/(t+q)}\Psi(h\xi^{(t+q)/\nu}). \tag{8}$$

As in the theory of second-order phase transitions, we have $\xi\sim|\tau|^{-\nu}$ (the critical exponents of course take on other numerical values.). Consequently (on the one hand), we find an expression for the smearing region, $h\xi^{(t+q)/\nu}\sim 1$, from (8):

$$\Delta\approx h^{1/(t+q)}. \tag{9}$$

In other words, the correlation length in the smearing region is

$$\xi_c=\xi(|\tau|\approx\Delta)\sim\Delta^{-\nu}\approx h^{-\nu/(t+q)}. \tag{10}$$

On the other hand, as before [see (6) and (7)], we have

$$\xi_c\sim h^{-\mu}=h^{-2/(d+2-\eta)}, \tag{11}$$

and thus

$$\frac{\nu}{t+q}=\frac{2}{s+2-\eta}. \tag{12}$$

Using the known relations between critical exponents, we can express the critical exponent η in terms of the familiar critical exponent β, which characterizes the density of a critical cluster: $\eta=2\beta/\nu-(d-2)$. We finally find

$$t+q=d\nu-\beta. \quad (13)$$

The combination $d\nu-\beta$ is well known in percolation theory: $d\nu-\beta=d_F\nu$, where d_F is the fractal dimensionality of the percolation cluster. The simultaneous use of the hypotheses of the scaling of the critical fluctuations in the theory of second-order phase transitions and the analogy between percolation-theory problems and the problem of a second-order phase transition thus leads to an equation which relates the "kinetic" critical exponents—the critical conductivity exponents t and q—to the fractal dimensionality of a percolation cluster, i.e., the geometric critical exponents.

In the 2D case we have[1,8] $t_2=q_2$ (the subscript specifies the dimensionality of the problem); from (13) we find

$$t_2=q_2=\nu_2-\frac{\beta_2}{2}. \quad (14)$$

This expression is the same as that derived for q_2 in Refs. 9 and 10. The values of ν_2 and β_2 are known quite exactly: $\nu_2=4/3$, $\beta_2=5/36$. We thus find the following result for the critical conductivity exponents in the 2D case:

$$t_2=q_2=\tfrac{91}{72}=1.263(8). \quad (15)$$

Numerically, relation (15) agrees well with values in the literature (see, for example, Ref. 12; the critical exponent t is denoted by μ in Table III). Relation (12) holds exactly in the $d=1$ case, for which we know (see, for example, Ref. 13) the exact values $\nu_1=1$, $\beta_1=0$, $q_1=1$, and also $t_1=0$.

Despite the exact agreement of relation (13) in the 1D case and the good numerical agreement in $d=2$ and 3 for the critical dimensionality in percolation, $d=d_c=6$, relation (13) is contradictory. It is generally believed (Ref. 13, for example) that we have $\nu_6=1/2$, $t_6=3$, and $q_6=0$. In order to satisfy (13) with $t_6=3$ and $\nu_6=1/2$, we would have to have $q_6=-1$. Consequently, relation (13) is not satisfied in $d=6$; therefore (if we assume $q_6=0$), there is a violation of the scaling hypothesis for critical fluctuations in percolation theory, at least in $d=6$.

On the other hand, calculations based on probability-theory considerations also yield $q_6^*=-1$. The result $t^*=1+\nu(d-2)$ was derived on the basis of these considerations in Ref. 7. In the case $d=d_c=6$, that relation leads to the exact value $t_6^*=t_6=1+(6-2)/2=3$. Corresponding considerations[14] for q yield $q^*=1-\nu(d-2)$; in the case $d=d_c=6$, this relation yields $q_6^*=1(6-2)/2=-1$, i.e., $q_6^*\neq q_6$. We note that the percolation threshold is zero in $d=6$. Consequently, the introduction of a critical exponent q to describe the behavior of the system to the left of the percolation threshold is problematic in this case.

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