Fluctuation limit of measurements of the relative elongation of a magnetostrictive cylinder

A. I. Golovashkin, A. V. Gudenko, L. N. Zherikhina, O. M. Ivanenko, K. Yu. Mitsen, and A. M. Tskhovrebov

P. N. Lebedev Physics Institute, Russian Academy of Sciences, 117024 Moscow, Russia

(Submitted 14 July 1994; resubmitted 15 September 1994)
Pis'ma Zh. Eksp. Teor. Fiz. **60**, No. 8, 595–599 (25 October 1994)

A method is proposed for detecting small changes in length by means of a SQUID–magnetostrictor system. By using an advanced SQUID with a resolution of $10^{-6}\Phi_0/\sqrt{\text{Hz}}$, one can develop a system for measuring relative elongations on the order of $10^{-21}/\sqrt{\text{Hz}}$ and less. This estimate demonstrates the potential for the use of SQUID–magnetorestrictor systems for detecting gravitational waves. Fluctuations of the magnetization of the magnetostrictor are incorporated in an analysis of the resolution of the system.

The idea of exploiting the inverse magnetostrictive effect in strain measurements is hardly new. There are detailed descriptions of working strain-measurement systems based on this principle in special reviews[1] and even in courses in general physics.[2]

In this letter we evaluate the limiting capabilities of this classical method in the case in which the magnetic response is detected by a superconducting quantum interferometer (SQUID). The resolution of an advanced commercial SQUID in terms of flux is limited by Nyquist noise and is usually about $3\times10^{-5}\Phi_0/\sqrt{\text{Hz}}=6.2\times10^{-20}$ Wb$/\sqrt{\text{Hz}}$, where $\Phi_0=\hbar/2e=2.07\times10^{-15}$ Wb is the quantum of magnetic flux.[3] A resolution on the order of $\delta\Phi=10^{-6}\Phi_0/\sqrt{\text{Hz}}=2.07\times10^{-21}$ Wb$/\sqrt{\text{Hz}}$ has been achieved in certain experiments.[4–6]

Let us estimate the relative elongation $\Delta L/L$ of a rod of a magnetostrictive material of cross-sectional area S, which causes a change in the magnetic flux penetrating the rod, $\Delta\Phi=\delta\Phi$, if the rod is in an external field H. The simplest characteristic of the inverse magnetorestrictive effect is the magnetostriction sensitivity constant Λ, which relates the change in the magnetic induction in the material to the stress which causes this change.[7] Ordinarily, Λ is on the order of (or greater than) 2×10^{-9} T/Pa. The stress which arises in a rod because of its elongation is found from Hooke's law. For an estimate we will use the typical value $E=200$ GPa for the Young's modulus of a solid. We then have $\delta\Phi=\Delta\Phi=S\Lambda E(\Delta L/L)$. Assuming $S=2\times10^{-2}$ m^2 ($=200$ cm^2), we find a minimal observable elongation $\delta L/L=2.5\times10^{-22}/\sqrt{\text{Hz}}$.

If the elongation $\delta L/L$ is to actually be detectable, the flux fluctuations $\delta\Phi_M$ generated by the magnetostrictive rod "itself" must be small in comparison with the Nyquist noise of the SQUID. Let us attempt to test this condition.

As an empirical upper estimate of the spectral density of the noise of a ferromagnetic core, we can use data from several studies of magnetic encephalography, in which biomagnetic fields are detected without the use of SQUIDs.[3,8]

In these experiments the total noise of the coil with the core at $T=300$ K ("room temperature") was 3×10^{-17} Wb/$\sqrt{\text{Hz}}$. It was asserted in Ref. 8 that this value was determined by the Nyquist noise of the coil resistance; i.e., it was asserted that the noise of the ferrite core was considerably weaker.

To find a theoretical estimate of $\delta\Phi_M$, we use the fluctuation dissipation theorem.[9] We write

$$\langle\delta\Phi_M^2\rangle=\frac{\hbar}{\pi}\int_0^\infty \alpha_M''(\omega)\coth(\hbar\omega/2kT)\,d\omega,$$

where $\alpha_M''(\omega)$ is the imaginary part of the generalized susceptibility of the magnetic system at the frequency ω, and the factor $\coth(\hbar\omega/2kT)=2\{[\exp(\hbar\omega/kT)-1]^{-1}+1/2\}$ reflects the Bose statistics of the "magnetic oscillators" and the contribution from zero-point vibrations. The generalized susceptibility of a system is expressed in terms of the magnetic susceptibility of the magnetorestrictive cylinder, $\chi(\omega)$, by $\alpha_M(\omega)=(S\mu_0/L)\chi(\omega)$, where μ_0 is the permeability of free space, and L and S are the length and cross-sectional area of the cylinder. The imaginary part of the magnetic susceptibility can be estimated by substituting experimental data on ferromagnetic resonance into the formula for $\chi''(\omega)$ found from a solution of the Bloch equations:[10]

$$\chi''(\omega)=\frac{\chi_0''\omega_0(B)\tau}{1+[\omega-\omega_0(B)]^2\tau^2}.$$

[The imaginary part of the generalized susceptibility turns out to be an odd function, $\alpha_M''(\omega,B)=-\alpha_M''(-\omega,-B)$, because of the change in the sign of the resonant frequency $\omega_0(B)$ upon a change in the sign of the magnetic field B.] For yttrium iron garnet in a field $B=0.11$ T, for example, we have $\chi_0''=1.3\times10^{-7}$, $\omega_0(B)/2\pi=3.3\times10^9$ Hz, and a transverse relaxation time $\tau=1.7\times10^{-6}/2\pi$ s. Determining the integrand in the formula which expresses the meaning of the fluctuation dissipation theorem in this manner, we find the frequency dependence of the spectral density of the noise generated by the rod:

$$\delta\Phi_M^2(\omega)=\frac{\mu_0\hbar S}{\pi L}\,\frac{\chi_0''\omega_0(B)\tau}{1+[\omega-\omega_0(B)]^2\tau^2}\coth(\hbar\omega/2kT).$$

Under the condition $kT\gg\hbar\omega$ we have $\coth(\hbar\omega/2kT)\to2kT/\hbar\omega$ and thus

$$\delta\Phi_M(\omega)=\left[\frac{2\mu_0SkT}{\pi L\omega}\,\frac{\chi_0''\omega_0(B)\tau}{1+[\omega-\omega_0(B)]^2\tau^2}\right]^{1/2}.$$

It follows from the last expression that in the limit $\omega\to0$ the frequency dependence of the spectral density is like the spectrum of the flicker noise (i.e., the $1/f$ noise).

For example, at a frequency of 1000 Hz in a band of 1 Hz, the noise generated by a magnetostrictive rod with dimensions $L=1$ m and $S=2\times10^{-2}$ m^2 at $T=4.2$ K ("liquid-helium temperature") is $\delta\Phi_M=1.5\times10^{-22}$ Wb/$\sqrt{\text{Hz}}$ according to the last formula. Comparing the value found for $\delta\Phi_M$ with the Nyquist noise of a SQUID, $\delta\Phi=2.07\times10^{-21}$ Wb/$\sqrt{\text{Hz}}$, we see that the fluctuations of the magnetic flux caused by the rod must be quite small and should not degrade the resolution of the system. The value found for $\delta\Phi_M$ above corresponds to transverse fluctuations of the flux, but a simple trigonometric conversion of the contribution of transverse fluctuations into longitudinal fluctuations (with which we should compare the Nyquist noise of a SQUID) shows that they are small even in comparison with $\delta\Phi_M$, to the extent that the ratio $\delta\Phi_M/\Phi$ is small, where Φ is the constant longitudinal magnetic flux. The scale of the fluctuations of the magnetic flux found from the Ginzburg–Landau theory agrees qualitatively with the estimate found above. It has also been shown that perturbations of the magnetic flux of a cylinder caused by thermodynamic fluctuations of the temperature (the magnon contribution)[9,11] are also quite small in the low-temperature region. The estimates above should of course be supplemented with direct measurements of the fluctuations of the magnetic flux in magnetostrictive materials at liquid-helium temperatures. Such measurements would make the formulation of an experiment of this sort realistic.

The detection system described schematically above is thus capable in principle of detecting a relative elongation at the level $2.5\times 10^{-22}/\sqrt{\text{Hz}}$. This figure is comparable in absolute value to the amplitude of the perturbation of the metric tensor $h=\sqrt{h_{yz}^2+h_{zz}^2}$ caused by the passage (in the X direction) of a gravitational wave with an intensity on the order of 10 erg/(cm$^2\cdot$s) (at a frequency $f=10^3$ Hz).[12] In principle, it thus becomes possible to use the system proposed here as a detector of gravitational waves of the indicated intensity. To some extent there may be an analogy here with the use of piezoelectric crystals to detect gravitational radiation.[13]

We might point out some particular features of measurements of the response of a magnetostrictive transducer. A magnetostrictive cylinder can be coupled with a SQUID in a natural way with the help of a superconducting flux transformer. When the transfer ratio of the flux transformer is taken into account, the overall sensitivity of the system in terms of flux may be slightly degraded. In Ref. 14, with a SQUID sensitivity $\delta\Phi=6\times 10^{-7}\Phi_0/\sqrt{\text{Hz}}$, the resolution referred to the input of the flux transformer was $2\times 10^{-5}\Phi_0/\sqrt{\text{Hz}}$. This figure corresponded to an optimum transfer ratio of 0.01 ($T=4.2$ K). By shielding the pickup loop of the transformer with an additional outer superconducting ring, however, one can effectively reduce the inductance of the pickup loop and thus increase the transfer ratio of the flux transformer. In the calculations, a value of one was used for the transfer ratio. This method has the advantage that the pickup loop can be near the cylinder without making mechanical contact with it, with no degradation of the overall Q of the system. The external field is introduced in an equally simple way by means of a small superconducting coil with frozen flux (after the cylinder is magnetized to its "working point," the external field can be turned off). If a piezomagnetic material (CoF_2 or MnF_2) is used as detector, one can do without external magnetization completely, although the cross-sectional area of the cylinder will have to be increased by about two orders of magnitude in order to preserve the sensitivity.

A characteristic feature of the detection system proposed here is that it is, to some extent, naturally immune to parasitic effects which are totally unrelated to the passage of the gravitational wave (and which are caused by, say, seismic vibrations). The magnetic response in such a system should arise only as a result of an elongation of the magnetostrictive cylinder. In a first approximation it is independent of a translational motion of the cylinder as a whole. There may also be a suppression of the parasitic signal corresponding to a rotation of the axis of the cylinder. The measured magnetic flux at the equilibrium position of the axis is at a maximum; since the first variation of the flux near its maximum has a zero value, the response of the system to a rotation of the axis is also zero in a first approximation.

The limitation on the resolution in terms of the relative elongation of "purely mechanical" origin in the case of a magnetostrictive transducer should not differ in a qualitative way from the corresponding restriction on the resolution of other gravitational detectors. Let us examine the limitation which is imposed by longitudinal elastic vibrations of a magnetostrictive cylinder. For this purpose we make use of the fluctuation dissipation theorem, which allows us to determine the spectral density of a noisy "elongation" of the cylinder, δL^2. For elastic vibrations, the frequency dependence of the imaginary part of the generalized susceptibility differs from the corresponding dependence of the magnetic fluctuations discussed above:

$$\alpha_L''(\omega)=\frac{8}{\pi^2 m}\frac{\omega\omega_n/Q_n}{(\omega^2-\omega_n^2)^2-(\omega\omega_n)^2/Q_n^2}.$$

In the limit $kT\gg\hbar\omega$, this frequency dependence of the susceptibility leads to an independence of the spectral density of the noisy elongation from the frequency (i.e., the noise turns out to be "white"). The intensity of the fluctuations of the relative elongation can then be written $(\delta L/L)^2=16kT/\pi^6 SQ_n V_s^3\rho n^3$, where m is the mass of the cylinder, ρ is the density of the magnetostrictive material, Q_n is the mechanical quality factor of the system, ω_n is the resonant frequency, n is the number of the overtone (an odd integer), and V_s is the longitudinal sound velocity. The limitation on the resolution at liquid-helium temperatures corresponding to a low overtone with a quality factor on the order of 10^3 is $\delta L/L=10^{-20}/\sqrt{\text{Hz}}$. This noise threshold could evidently be reduced substantially by increasing, to the extent possible, the quality factor Q_n and the cross-sectional area of the cylinder, S (in a numerical estimate, we adopted $S=2\times10^{-2}$ m^{-2}, $\rho=8\times10^3$ kg/m^3, and $V_s=5\times10^3$ m/s). We would expect a greater effect from the increase in S, since in this case the decrease in $(\delta L/L)^2$ would be accompanied by an increase in the "sensitivity" $[\Delta\Phi=S\Lambda E(\Delta L/L)]$. An increase in the sensitivity also (among other things) "cancels" the increase in the magnetic fluctuations $\delta\Phi_{\text{M}}^2(\omega)$ proportional to S, so the magnetic noisy elongation $\delta L_{\text{M}}^2/L^2=\delta\Phi_{\text{M}}^2(\omega)/S^2\Lambda^2E^2$ decreases with increasing S as $1/S$.

We conclude with one more particular feature of a magnetostrictive detector. As can be seen from the equations written above, fluctuations $\delta L/L$ do not depend on the length of the cylinder, L, and the amplitude of the magnetic fluctuations discussed above, $\delta\Phi_M(\omega)$, is proportional to $1/\sqrt{L}$. In other words, it depends only weakly on L. Since the sensitivity of the detection system proposed here does not depend on the length of the cylinder $[\Delta\Phi=S\Lambda E(\Delta L/L)]$, we conclude that we have some latitude in the choice of L. There can be no such latitude in the case of a piezoelectric detector, since its sensitivity is proportional to the linear dimension $(\Delta U=(LdE/\epsilon_0\epsilon)(\Delta L/L)$, where ΔU is the potential difference generated by the pickup, d is the piezoelectric modulus, and $\epsilon_0\epsilon$ is the dielectric constant). This "latitude" in the choice of L means that one can develop a compact magnetostrictive detector of gravitational waves. At the same time, we should point out that in estimating the sensitivity of the system we used "optimistic" values of such parameters as the transfer ratio of the flux transformer and the sensitivity of the SQUID. Achieving these optimistic values in a real situation would not be a simple matter.

[1] K. P. Belov, *Magnetostrictive Phenomena and Their Technical Applications* [in Russian] (Nauka, Moscow, 1987).
[2] K. A. Putilov, *Course in Physics* [in Russian] (Fiz.-Mat. Lit., Moscow, 1963), p. 2.
[3] *Weak Superconductivity* [Russian translation], ed. by B. B. Shvarts and S. Foner (Mir, Moscow, 1980).
[4] R. F. Voss *et al.*, J. Appl. Phys. **51**, 2306 (1980).
[5] M. B. Ketchen and F. Voss, Appl. Phys. Lett. **35**, 812 (1979).
[6] J. Clarke, IEEE Trans. Electron Dev. **ED-27**, 1396 (1980).
[7] G. M. Sverdlin, *Applied Hydroacoustics* [in Russian] (Sudostroenie, Leningrad, 1990).
[8] D. Cohen, Science **161**, 784 (1968).
[9] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980).
[10] N. M. Pomerantsev *et al.*, *Physics of Quantum Magnetometry* [in Russian] (Fiz.-Mat. Lit., Moscow, 1972).
[11] E. M. Lifshitz and L. P. Pitaevskiĭ, *Statistical Physics* [in Russian] (Fiz.-Mat. Lit., Moscow, 1978).
[12] I. Bichak and V. N. Rudenko, *Gravitational Waves and the Problem of Detecting Them* (MGU, Moscow, 1987).
[13] J. Weber, Phys. Rev. **117**, 306 (1960).
[14] M. B. Ketchen and J.M.J. Aycox, Appl. Phys. Lett. **40**, 736 (1982).

Translated by D. Parsons