Level statistics in a metallic sample: corrections to the Wigner–Dyson distribution

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The deviation of the level correlation function in a mesoscopic metallic sample from the Wigner–Dyson distribution is calculated by using a combination of the renormalization group and nonperturbative treatment. For a given spatial dimension the correction is determined by the sample conductance. © *1994 American Institute of Physics.*

The problem of level correlation in quantum systems has attracted interest of physicists since the work of Wigner.[1] The random matrix theory developed by Wigner and Dyson[2] describes well the level statistics of various complex systems like nuclei or atoms. Later, Gor'kov and Eliashberg[3] put forward an assumption that the random matrix theory is also applicable to the problem of energy level correlations of a quantum particle in a random potential. This hypothesis was proven by Efetov, who showed[4] that the level–level correlation function $R(\omega)$ (its formal definition is given below) is described by the Wigner–Dyson distribution for $\omega \ll E_c$, where E_c is the Thouless energy. For $\omega \gg E_c$, the behavior of the correlation function changes because the corresponding time scale ω^{-1} is smaller than the time E_c^{-1} the particle needs to diffuse through the sample. The form of the correlation function in this region which was calculated in Ref. 5 by means of the diffusion perturbation theory, depends on the spatial dimensionality.

In the present letter we find a correction to the Wigner–Dyson distribution in the region $\omega \sim \Delta \ll E_c$, where Δ is the mean level spacing. This is not a trivial task, because we calculate a correction to the result which is essentially nonperturbative.

We study the two-level correlation function $R(s)$ defined as

$$R(s)=\frac{1}{\langle \nu \rangle^2}\langle \nu(E)\nu(E+\omega)\rangle, \tag{1}$$

where $s=\omega/\Delta$, $\nu(E)$ is the density of states at the energy E, and $\langle ... \rangle$ denotes averaging over realizations of the random potential. As was shown by Efetov,[4] the correlator (1) can be expressed in terms of the Green's function for a certain supermatrix σ-model. Depending on whether the time reversal and spin rotation symmetries are broken, one of the three different σ-models is relevant, with a unitary, orthogonal, or symplectic symmetry group. We will consider the case of the unitary symmetry (which corresponds to the broken time-reversal invariance) throughout the paper; the results for two other cases will be presented at the end. The expression for $R(s)$ in terms of the σ-model then reads

$$R(s)=\left(\frac{1}{4V}\right)^2 \text{Re} \int DQ(r)\left[\int d^dr \text{ Str } Q\Lambda k\right]^2$$

$$\times\exp\left\{-\frac{\pi\nu}{4}\int d^dr \text{ Str}[-D(\nabla Q)^2-2i\omega\Lambda Q]\right\}. \quad (2)$$

Here $Q=T^{-1}\Lambda T$ is 4×4 supermatrix, with T belonging to the coset space $U(1,1|2)$, $\Lambda=\text{diag}\{1,1,-1,-1\}$, $k=\text{diag}\{1,-1,1,-1\}$, Str denotes the supertrace, V is the system volume, and D is the classical diffusion constant. To find a detailed description of the model and mathematical entities involved, a reader is referred to Refs. 4 and 6.

For $\omega\ll E_c\sim D/L^2$ (L is the system size, so that $V=L^d$) the leading contribution to the integral (2) is given by the spatially uniform fields $Q(r)=Q$. Then the functional integral in Eq. (2) reduces to an integral over a single supermatrix Q and can be calculated yielding the Wigner–Dyson distribution:[4]

$$R_{\text{WD}}(s)=1-\frac{\sin^2(\pi s)}{(s)^2}. \quad (3)$$

The aim of the present paper is to calculate a correction to Eq. (3) due to the spatial fluctuations of $Q(r)$ in Eq. (2). The procedure we are using for this purpose is as follows. We first decompose Q into the constant part Q_0 and the contribution $\tilde{Q}$ of higher modes with nonzero momenta. We then use the renormalization group ideas and integrate over all fast modes. This can be done *perturbatively* provided the dimensionless (measured in units of $e^2/\hbar$) conductance $g=E_c/\Delta\gg1$. As a result, we obtain an integral over the matrix Q_0 only, which must be calculated *nonperturbatively*. We believe that this combination of the perturbative renormalization-group-type and nonperturbative treatments is of a methodological interest and might be used for other applications.

First, we present the correlator $R(s)$ in the form

$$R(s)=\frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial u^2}\int DQ \exp\{-\mathcal{F}(Q)\}|_{u=0},$$

$$\mathcal{F}(Q)=-\frac{1}{t}\int \text{Str}(\nabla Q)^2+\tilde{s}\int \text{Str}\Lambda Q+\tilde{u}\int \text{Str } Q\Lambda k, \quad (4)$$

where $1/t=\pi\nu D/4$, $\tilde{s}=\pi s/2iV$, and $\tilde{u}=\pi u/2iV$. We then decompose Q in the following way:

$$Q(r)=T_0^{-1}\tilde{Q}(r)T_0, \quad (5)$$

where T_0 is a spatially uniform matrix, and $\tilde{Q}$ describes all modes with nonzero momenta. When $\omega\ll E_c$, the matrix Q fluctuates only weakly near the origin Λ of the coset space. In the leading order $\tilde{Q}=\Lambda$; thus (4) are reduced to zero-dimensional σ-model, which leads to the Wigner–Dyson distribution (3). To find the corrections, we should expand the matrix $\tilde{Q}$ around the origin Λ. This expansion starts as[4]

$$\tilde{Q}=\Lambda\left(1+W+\frac{W^2}{2}+\cdots\right), \tag{6}$$

where W is a supermatrix with the following block structure:

$$W=\begin{pmatrix} 0 & t_{12} \\ t_{21} & 0 \end{pmatrix}. \tag{7}$$

Substituting this expansion into Eq. (4), we obtain

$$\mathcal{F}=\mathcal{F}_0+\mathcal{F}_1+O(W^3),$$

$$\mathcal{F}_0=\int \mathrm{Str}\left[\frac{1}{t}(\nabla W)^2+\tilde{s}Q_0\Lambda+\tilde{u}Q_0\Lambda k\right],$$

$$\mathcal{F}_1=\tfrac{1}{2}\int \mathrm{Str}[\tilde{s}U_0\Lambda W^2+\tilde{u}U_{0k}\Lambda W^2], \tag{8}$$

where $Q_0=T_0^{-1}\Lambda T_0$, $U_0=T_0\Lambda T_0^{-1}$, and $U_{0k}=T_0\Lambda k T_0^{-1}$. We define $\mathcal{F}_{\mathrm{eff}}(Q_0)$ as a result of elimination of the fast modes:

$$e^{-\mathcal{F}_{\mathrm{eff}}(Q_0)}=e^{-\mathcal{F}_{\mathrm{eff}}(Q_0)}\langle e^{-\mathcal{F}_1(Q_0,W)}\rangle_W, \tag{9}$$

where $\langle\cdots\rangle_W$ denote the integration over W. Expanding to the order W^4, we obtain

$$\mathcal{F}_{\mathrm{eff}}=\mathcal{F}_0+\langle\mathcal{F}_1\rangle-\tfrac{1}{2}\langle\mathcal{F}_1^2\rangle+\tfrac{1}{2}\langle\mathcal{F}_1\rangle^2+\cdots. \tag{10}$$

The integral over the fast modes can now be calculated by using the Wick theorem and the contraction rules:[4,7]

$$\langle \mathrm{Str}\, W(r)PW(r')R\rangle=\Pi(r-r')\mathrm{Str}\, P\, \mathrm{Str}\, R-\mathrm{Str}\, P\Lambda\, \mathrm{Str}\, R\Lambda),$$

$$\langle \mathrm{Str}[W(r)P]\mathrm{Str}[W(r')R]\rangle=\Pi(r-r')\mathrm{Str}(PR-P\Lambda R\Lambda),$$

$$\Pi(r)=\int \frac{d^dq}{(2\pi)^d}\frac{\exp(iqr)}{\pi\nu Dq^2}, \tag{11}$$

where P and R are arbitrary supermatrices. The result is

$$\langle\mathcal{F}_1\rangle=0,$$

$$\langle\mathcal{F}_1^2\rangle=\tfrac{1}{2}\int drdr'\Pi^2(r-r')(\tilde{s}\,\mathrm{Str}\, Q_0\Lambda+\tilde{u}\,\mathrm{Str}\, Q_0\Lambda k)^2. \tag{12}$$

Substituting Eq. (12) into Eq. (10), we find

$$\mathcal{F}_{\mathrm{eff}}(Q_0)=\frac{\pi}{2i}s\,\mathrm{Str}\, Q_0\Lambda+\frac{\pi}{2i}u\,\mathrm{Str}\, Q_0\Lambda k+\frac{a_d}{16g^2}(s\,\mathrm{Str}\, Q_0\Lambda+u\,\mathrm{Str}\, Q_0\Lambda k)^2,$$

$$a_d=\frac{1}{\pi^4}\sum_{\substack{n_1,\ldots,n_d=0\\ n_1^2+\cdots+n_d^2>0}}^{\infty}\frac{1}{(n_1^2+\cdots+n_d^2)^2}. \tag{13}$$

The value of the coefficient a_d depends on spatial dimensionality d. In particular, for $d=1,2,3$ we have $a_1=1/90\simeq 0.0111$, $a_2\simeq 0.0266$, and $a_3\simeq 0.0527$, respectively. Using now Eq. (4), we find the following expression for the correlator to the $1/g^2$ order:

$$R(s)=\mathrm{Re}\,\frac{1}{(2\pi i)^2}\int dQ_0\left\{\left(\frac{\pi}{2i}\right)^2(\mathrm{Str}\,Q_0\Lambda k)^2\left[1-\frac{a_d}{16g^2}s^2(\mathrm{Str}\,Q_0\Lambda)^2\right]\right.$$
$$\left.-\frac{a_d}{8g^2}[\mathrm{Str}\,Q_0\Lambda k)\Big]^2+\frac{\pi a_d}{8g^2 i}s(\mathrm{Str}\,Q_0\Lambda)(\mathrm{Str}\,Q_0\Lambda k)^2\right\}\exp\left\{-\frac{\pi}{2i}s\,\mathrm{Str}Q_0\Lambda\right\}. \tag{14}$$

This integral over the supermatrix Q_0 can be calculated by using the known technique,[4] yielding

$$R(s)=1-\frac{\sin^2(\pi s)}{(\pi s)^2}+\frac{a_d}{\pi^2 g^2}\sin^2(\pi s). \tag{15}$$

The last term in Eq. (15) represents the correction of order $1/g^2$ to the Wigner–Dyson distribution. Equation (15) is valid for $s\ll g$. Let us note that the smooth (nonoscillating) part of this correction in the region $1\ll s\ll g$ can be found by using a purely perturbative approach.[5,8] For $s\gg 1$ the leading perturbative contribution to $R(s)$ is given by a two-diffusion diagram:

$$R_{\mathrm{pert}}(s)-1=\frac{\Delta^2}{2\pi^2}\,\mathrm{Re}\sum_{\substack{q_i=\pi n_i/L\\ n_i=0,1,2,\ldots}}\frac{1}{(Dq^2-i\omega)^2}=\frac{1}{2\pi^2}\,\mathrm{Re}\sum_{n_i\geq 0}\frac{1}{(-is+\pi^2 g n^2)^2}\,. \tag{16}$$

At $s\ll g$, this expression is dominated by the $q=0$ term, with other terms giving a correction of order $1/g^2$:

$$R_{\mathrm{pert}}(s)=1-\frac{1}{2\pi^2 s^2}+\frac{a_d}{2\pi^2 g^2}\,, \tag{17}$$

where a_d was defined in Eq. (13). This formula, obtained in the region $1\ll s\ll g$, is perturbative in both $1/s$ and $1/g$. It does not contain oscillations (which cannot be found perturbatively) and gives no information about the actual small-s behavior of $R(s)$. The result (15) of the present letter is much stronger: it represents the exact (nonperturbative in $1/s$) form of the correction in the whole region $s\ll g$.

The calculation presented above can be straightforwardly generalized to the other symmetry cases. The result can be presented in a form valid for all three cases:

$$R_\beta(s)=\left(1+\frac{a_d}{2\beta\pi^2 g^2}\frac{d^2}{ds^2}s^2\right)R_\beta^{WD}(s), \tag{18}$$

$\beta=1(2,4)$ for the case of orthogonal (unitary, symplectic) symmetry; R_β^{WD} denotes the corresponding Wigner–Dyson distribution, whose explicit form can be found, e.g., in Refs. 4 and 9. (We denote by g the conductance per spin projection: $g=E_c/\Delta=\nu DL^{d-2}$, without multiplication by a factor of 2 due to the spin.)

In the limit $s\to 0$, the Wigner–Dyson distribution has the behavior

$$R_\beta^{WD} \simeq c_\beta s^\beta, \quad s \to 0,$$

$$c_1 = \frac{\pi^2}{6}, \quad c_2 = \frac{\pi^2}{3}, \quad c_4 = \frac{\pi^4}{135}. \tag{19}$$

As is clear from Eq. (18), this correction does not change the power β, but renormalizes the prefactor c_β:

$$R_\beta(s) = \left(1 + \frac{(\beta+2)(\beta+1)}{2\beta} \frac{a_d}{\pi^2 g^2}\right) c_\beta s^\beta, \quad s \to 0. \tag{20}$$

The correction to c_β is positive. Physically, this means a weakening of the level repulsion.

In conclusion, we have calculated the deviation of the level-level correlation function $R_\beta(s)$ in a mesoscopic metallic sample from its universal Wigner–Dyson form, using the supersymmetric sigma-model approach. The resulting correction is of order $1/g^2$, where g is the dimensionless conductance. It does not change the power β of the s^β behavior of the correlator $R(s)$ as $s \to 0$, but renormalizes the corresponding prefactor.

To obtain this result, we developed a novel calculation method, which combines perturbative elimination of fast diffusive modes (in the spirit of renormalization-group ideas) and the subsequent nonperturbative evaluation of an integral over the zero mode. In the present paper we have used it to find the eigenvalue correlation function, but it may have other applications. In particular, deviation of eigenfunctions statistics in diffusive regime from the random matrix theory predictions can be studied in this manner.[10]

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