Pionium in decays of elementary particles

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Atomic decays of elementary particles which give rise to pionium, i.e., an atom consisting of positively and negatively charged pions, are discussed. The relativistic correction to the lifetime of pionium is calculated. The possibility of studying these decays at meson factories is discussed. © *1994 American Institute of Physics.*

Pionium (the dimesic $\pi^+\pi^-$ atom) was recently discovered experimentally.[1] Although its lifetime has not been measured, it presumably will be in further experiments.[2] Measurements of this sort might provide a model-independent way to find information on the scattering length of pions.[3,4] Although pionium was first discussed in Ref. 3, the current experimental research on atoms of this sort was initiated by papers by Nemenov.[5] Additional information of the properties of such atoms can be found in Refs. 6 and 7.

Even the first papers on pionia discussed the possibility that they would form in decays of elementary particles. Since the corresponding widths are very small, research on such decays has not been worthwhile until now, when several high-luminosity meson factories are expected to come on line. Below we discuss certain atomic decays involving pionium in the final state.

In the nonrelativistic approximation, the amplitude for the atomic decay, $M_1 \to M_2 + A_{2\pi}$ is given by[8]

$$\langle M_2 A_{2\pi} | M_1 \rangle = I\left(Q_1, \frac{P_A}{2}, \frac{P_A}{2}, Q_2 \right) \frac{i\Psi(x=0)}{\sqrt{m}}, \tag{1}$$

where $m = m_{\pi^+}$, $\Psi(x=0)$ is the Schrödinger wave function of the hydrogen-like atom at the origin, and $I(Q_1, p_+, p_-, Q_2)$ is the amplitude for the nonatomic decay $M_1(Q_1) \to \pi^+(p_+) + \pi^-(p_-) + M_2(Q_2)$. If this amplitude is known, expression (1) can be used to calculate the ratio of widths for atomic and nonatomic decays. For example, using the experimental Dalitz plot for[9] $\eta \to \pi^+\pi^-\pi^0$,

$$|I(x,y)|^2 \sim 1 - (1.08 \pm 0.014)y + (0.03 \pm 0.03)y^2 + (0.05 \pm 0.03)x^2, \tag{2}$$

we find

$$\frac{\Gamma(\eta \to \pi^0 A_{2\pi})}{\Gamma(\eta \to \pi^+\pi^-\pi^0)} \approx 0.91 \times 10^{-7}. \tag{3}$$

(We are assuming here that $A_{2\pi}$ forms in the $1S$ state. If a sum is taken over all nS states, the result is increased by a factor of about 1.2). The value of 3.9×10^{-7} cited in Ref. 7

corresponds to the theoretical predication[10] $I(x,y) \sim 1-0.55y$ and may prove too optimistic, although the errors in the determination of the quadratic terms in (2) are such that we cannot rule out the possibility of a several fold increase in the value in (3).

Using the experimental data analogous to (2) for $K^+ \to \pi^+ + \pi^+ \pi^-$ (Ref. 11), $K_L \to \pi^+ \pi^- \pi^0$ (Ref. 12) and $\eta' \to \eta \pi^+ \pi^-$ (Ref. 13), we find

$$\frac{\Gamma(K^+ \to \pi^+ A_{2\pi})}{\Gamma(K^+ \to \pi^+ \pi^- \pi^0)} \approx 10^{-5},$$

$$\frac{\Gamma(K_L \to \pi^0 A_{2\pi})}{\Gamma(K_L \to \pi^+ \pi^- \pi^0)} \approx 8.6 \times 10^{-7}, \quad \frac{\Gamma(\eta' \to \eta A_{2\pi})}{\Gamma(\eta' \to \eta \pi^+ \pi^-)} \approx 1.4 \times 10^{-6}. \tag{4}$$

In the case of $K^+ \to \pi^+ A_{2\pi}$ decay, we need to allow for the circumstance that the π^+ mesons are identical; when we do this, the result is doubled. Another difference of a factor of about 5 between K^+ and K_L decays is attributed to different shapes of the Dalitz plot for the decays $K^+ \to \pi^+ \pi^+ \pi^-$ and $K_L \to \pi^+ \pi^- \pi^0$. The decay $K^+ \to \pi^+ A_{2\pi}$ was discussed in Ref. 14.

For $c-\tau$ and B-meson factories, the atomic decays $\psi(2S) \to \psi(1S) A_{2\pi}$ and $\Upsilon(2S) \to \Upsilon(1S) A_{2\pi}$ may be interesting, since the widths of the corresponding nonatomic decays are larger. Taking information on the amplitudes of nonatomic decays from Ref. 15, we find

$$\frac{\Gamma[\psi(2S) \to \psi(1S) A_{2\pi}]}{\gamma[\psi(2S) \to \psi(1S) \pi^+ \pi^-]} \approx 4.6 \times 10^{-8},$$

$$\frac{\Gamma[\Upsilon(2S) \to \Upsilon(1S) A_{2\pi}]}{\Gamma[\Upsilon(2S) \to \Upsilon(1S) \pi^+ \pi^-]} \approx 5.2 \times 10^{-8}. \tag{5}$$

The results in (3)–(5) lead to the following probabilities for atomic decays:

$$Br[\psi(2S) \to \psi(1S) A_{2\pi}] \approx 1.4 \times 10^{-8}, \quad Br[\Upsilon(2S) \to \Upsilon(1S) A_{2\pi}] \approx 10^{-8},$$

$$Br(\eta \to \pi^0 A_{2\pi}) \approx 2 \times 10^{-8}, \quad Br(\eta' \to \eta A_{2\pi}) \approx 6.2 \times 10^{-7},$$

$$Br(K^+ \to \pi^+ A_{2\pi}) \approx 5.5 \times 10^{-7}, \quad Br(K_L \to \pi^0 A_{2\pi}) \approx 1.1 \times 10^{-7}. \tag{6}$$

For a B factory, these figures correspond to a few events per year, so it would be unrealistic to study the atomic decays of Υ. For a φ factory, in contrast, we would expect $\sim 10^4$ atomic decays of K mesons involving pionium per year. An experimental study of such decays would thus be completely realistic.

We turn now to the corrections $O(\alpha)$ to the lifetime of pionium. The amplitude for the primary decay of pionium, $A_{2\pi} \to \pi^0 \pi^0$, is given by the expression[16]

$$\langle \pi^0 \pi^0 | A_{2\pi} \rangle = \int \frac{dp}{(2\pi)^4} J\left(p_1, p_2, \frac{P_A}{2}+p, \frac{P_A}{2}-p\right) \chi(p), \tag{7}$$

where $\chi(p)$ is the Bethe–Salpeter wave function for the $A_{2\pi}$ bound state, and $J(p_1, p_2, p_+, p_-)$ is the $\pi^+ \pi^-$ -irreducible kernel for the transition $\pi^+(p_+) + \pi^-(p_-) \to \pi^0(p_1) + \pi^0(p_2)$. We are using the relativistic normalization

$$\langle \mathbf{p}|\mathbf{q}\rangle 2E_p(2\pi)^3\delta(\mathbf{p}-\mathbf{q}).$$

At $O(\alpha)$, the quantity J is constant and is determined by the pion scattering lengths a_0, and a_2 (Ref. 17):

$$J=\frac{32}{3}\pi m(a_0-a_2). \tag{8}$$

We can thus rewrite (7) as

$$\langle \pi^0\pi^0|A_{2\pi}\rangle = J\times\chi(x=0), \tag{9}$$

where

$$\chi(x=0)=\int \frac{dp}{(2\pi)^4}\chi(p) \tag{10}$$

is the Bethe–Salpeter wave function at the origin.

The Bethe–Salpeter equation for $\chi(p)$ at $O(\alpha)$ is (in the rest frame of the $A_{2\pi}$, with $\lambda=e^2M_A^2/16\pi^2$)

$$\left[m^2+\mathbf{p}^2-\left(\frac{M_A}{2}+p_0\right)^2\right]\left[m^2+\mathbf{p}^2-\left(\frac{M_A}{2}-p_0\right)^2\right]\chi(p)=\frac{i\lambda}{\pi^2}\int dq\,\frac{\chi(q)}{(p-q)^2-i\epsilon}. \tag{11}$$

It corresponds to the Wick–Cutkosky model.[18] This circumstances was first noted in Ref. 19 and was utilized there to calculate a correction $O(\alpha)$ to the width of the decay $K_L\rightarrow \nu A_{\mu\pi}$.

According to Ref. 18, the solution of (11) for the ground state (a 1S state in the nonrelativistic limit) is

$$\chi(p)=\int_{-1}^{1}\frac{g(z)dz}{[A+Bz]^3}, \tag{12}$$

where

$$A=m^2-\tfrac{1}{4}M_A^2-p^2\equiv\Delta^2-p^2,\quad B=p_0M_A, \tag{13}$$

and the spectral function $g(z)$ is given by the integral equation

$$g(z)=\frac{\lambda}{2}\int_{-1}^{1}\frac{1}{\Delta^2+\frac{1}{4}M_A^2y^2}\left[\frac{1-z}{1-y}\Theta(z-y)+\frac{1+z}{1+y}\Theta(y-z)\right]g(y)dy. \tag{14}$$

However, we have $\Delta^2=m^2-\frac{1}{4}M_A^2\approx\frac{1}{4}m^2\alpha^2$, so we can write

$$\frac{1}{\Delta^2+\frac{1}{4}M_A^2y^2}=\frac{2\pi}{m^2\alpha}\delta(y)+\sigma(y), \tag{15}$$

where $\sigma(y)$ is a small quantity $O(\alpha)$ in comparison with the first term $\sim\delta(y)$. Taking $g(z)=g_0(z)+\alpha g_1(z)$, we find from (14) and (15) (N_0 and N_1 are constants)

$$g_0(z)=N_0(1-|z|),\quad g_1(z)=N_1(1-|z|)+\frac{\lambda}{2\alpha}\int_{-1}^{1}\sigma(y)R(z,y)g_0(y)dy, \tag{16}$$

where

$$R(z,y)=\frac{1-z}{1-y}\,\Theta(z-y)+\frac{1+z}{1+y}\,\Theta(y-z).$$

Evaluating the integral in (16) in the limit $\alpha\to 0$, and noting that we have $\lambda\pi/m^2\alpha=1$ for the ground state, we find

$$g_1(z)=N_1(1-|z|)+\frac{N_0}{\pi}\{(1-|z|)\ln(\alpha)+(1+|z|)[\ln(2|z|)-\ln(1+|z|)]\}.$$

An an accuracy $O(\alpha)$ we thus have

$$g(z)=N\left\{(1-|z|)+\frac{\alpha}{\pi}(1+|z|)[\ln(2|z|)-\ln(1+|z|)]\right\}, \tag{17}$$

from which we find the following expression for the Bethe–Salpeter wave function in (12):

$$\chi(p,P_A)=\frac{N}{(\Delta^2-p^2)\left[m^2-\left(\frac{P_A}{2}+p\right)\right]^2\left[m^2-\left(\frac{P_A}{2}-p\right)^2\right]}\left\{1+\frac{\alpha}{\pi}\chi_1(p,P_A)\right\}. \tag{18}$$

Here

$$\chi_1(p,P_A)=\frac{m^2-\left(\frac{P_A}{2}-p\right)^2}{2(\Delta^2-p^2)}\ln\left[m^2-\left(\frac{P_A}{2}-p\right)^2\right]+\frac{m^2-\left(\frac{P_A}{2}+p\right)^2}{2(\Delta^2-p^2)}\ln\left[m^2-\left(\frac{P_A}{2}+p\right)^2\right]-\ln(\Delta^2-p^2)+O[\alpha\ln(\alpha)]. \tag{19}$$

The normalization constant N is found from the normalization condition of Ref. 20. It can be shown to be

$$N=32\sqrt{\pi m}(\tfrac{1}{2}m\alpha)^{5/2}\left(1+\frac{\alpha}{\pi}\right). \tag{20}$$

In the nonrelativistic limit, (18) and (20) lead to

$$\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\,\chi(p)=\frac{i}{\sqrt{m}}\,\psi(\mathbf{p}), \tag{21}$$

where $\psi(\mathbf{p})$ is the Schrödinger wave function in momentum space. This circumstance explains the presence of the factor $i/\sqrt{m}$ in Eq. (1).

Substituting the value found for $\chi(p)$ into (10), and evaluating the integral to within terms $O(\alpha)$, we find

$$\chi(x=0)\approx\frac{i}{\sqrt{m}}\Psi(x=0)\left(1+2\frac{\alpha}{\pi}\right). \tag{22}$$

For the width of the $A_{2\pi}\to\pi^0\pi^0$ decay this expression yields the following correction $O(\alpha)$:

$$\Gamma(A_{2\pi}\to\pi^0\pi^0)=\Gamma_0(A_{2\pi}\to\pi^0\pi^0)\left(1+4\frac{\alpha}{\pi}\right). \tag{23}$$

An expression for Γ_0 is given in Ref. 4, among other places.

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[1] L.G. Afanasyev *et al.*, Phys. Lett. B **308**, 200 (1993).
[2] Bern–Dubna Collab., "Lifetime measurement of $\pi^+\pi^-$ atoms to test low energy QCD predictions" (letter of intent), SPSLC 92-44/I191; H. Nann, in: *Proc. Workshop on Meson Production, Interaction, and Decay* (Krakow, Poland, 1991), ed. by A. Magiera, W. Oelert, and E. Grosse (World Scientific, Singapore, 1991), p. 100.
[3] J. L. Uretsky and T. R. Palfrey, Phys. Rev. **121**, 1798 (1961).
[4] S. M. Bilen'kiĭ *et al.*, Yad. Fiz. **10**, 812 (1969) [Sov. J. Nucl. Phys. **10**, 469 (1969)].
[5] L. L. Nemenov, Yad. Fiz. **15**, 1047 (1972) [Sov. J. Nucl. Phys. **15**, 582 (1972)]; L. L. Nemenov, Yad. Fiz. **16**, 125 (1972) [Sov. J. Nucl. Phys. **16**, 67 (1972)].
[6] L. L. Nemenov, Yad. Fiz. **41**, 980 (1985) [Sov. J. Nucl. Phys. **41**, 629 (1985)]; A. A. Bel'kov *et al.*, Yad. Fiz. **44**, 466 (1986) [Sov. J. Nucl. Phys. **44**, 300 (1986)]; G. V. Efimov *et al.*, Yad. Fiz. **44**, 460 (1986) [Sov. J. Nucl. Phys. **44**, 296 (1986)].
[7] S. Wycech and A. M. Green, Nucl. Phys. A **562**, 446 (1993).
[8] R. Staffin, Phys. Rev. D **16**, 726 (1977).
[9] J. G. Layter *et al.*, Phys. Rev. D **7**, 2565 (1973).
[10] S. Fajfer and J.-M. Gerard, Z. Phys. C **42**, 431 (1989).
[11] B. Devaux *et al.*, Nucl. Phys. B **126**, 11 (1977).
[12] R. Messner *et al.*, Phys. Rev. Lett. **33**, 1458 (1974).
[13] G. R. Kalbfleish, Phys. Rev. D **10**, 916 (1974).
[14] H. Pilkuhn and S. Wycech, Phys. Lett. B **76**, 29 (1978).
[15] H. Albrecht *et al.*, Z. Phys. C **35**, 283 (1987).
[16] S. Mandelstam, Proc. R. Soc. **233**, 248 (1955).
[17] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
[18] G. C. Wick, Phys. Rev. **96**, 1124 (1954); R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).
[19] C. Ching *et al.*, Phys. Lett. B **98**, 456 (1981).
[20] N. Nakanishi, Phys. Rev. B **138**, B1182 (1965).

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