Exact solution for a layered two-dimensional Ising lattice

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The free energy is calculated for the Ising model with three independent coupling constants I_1, I_2, and I_3, respectively, along the columns and rows with numbers of different parity. The low-temperature properties of the model for different ratios of I_1, I_2, and I_3 and the existence conditions of the phase transition are analyzed.

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The question of the existence of a phase transition in spin glasses has aroused interest in systems containing frustrations. One such system is the odd Ising model[1] that can be solved exactly. A generalization of the odd model, called the "piled-up dominos" model by the authors,[2] was examined in Ref. 2. In this model the coupling constants in the columns (I_1) and in the rows with numbers of the same parity (I_2) are equal to each other: $I_1 = I_2 = I$, but the coupling constant in rows with numbers having the other parity (I_3) can differ from I.

In this paper we present the results for a layered lattice with three arbitrary constants: I_1 in the columns, I_2 and I_3 in the rows. It is shown that there is no phase transition for any I_1 only if $I_2 = I_3$, and the low-temperature properties of the model, which are strongly dependent on the ratios of I_1, I_2, and I_3, are examined. It is easy to see that by means of the Mattis transformation (simultaneous change of the signs of some constants and spin variables in the sublattice) any system of the described type can be reduced to a lattice whose coupling constants satisfy the following conditions:

$$I_1 \geqslant 0; \; I_2 \geqslant |I_3| \geqslant 0. \tag{1}$$

Below, the inequalities (1) are assumed to be satisfied. To calculate the free energy, like in Ref. 2, we used the transition matrix method.[3] The largest eigenvalue of the transition matrix is proportional to $\exp\{\frac{1}{2}\Sigma_p \epsilon_p\}$

$$\begin{aligned}\cosh \epsilon_p &= \cosh 2(K_2 + K_3) - \sinh 2(K_2 + K_3)\sinh 4\tilde{K}_1 \cos p \\ &\quad - 2\,\mathrm{sh}^2 2\tilde{K}_1\,\mathrm{sh}\,2K_2\,\mathrm{sh}\,2K_3 \sin^2 p.\end{aligned} \tag{2}$$

Here $K_i = I_i\beta$ ($i = 1, 2, 3$), $\tanh \tilde{K}_1 = e^{-2K_1}$, $0 \leqslant p \leqslant \pi$. The quantity ϵ_p vanishes at only one point (0 or π) at the temperature $T_c = 1/\beta_c$, defined by the condition

$$\sinh 2I_1\beta_c \sinh|I_2 + I_3|\beta_c = 1. \tag{3}$$

It follows from (3) that as $I_2 + I_3 \to 0$, the phase transition temperature $T_c \to 0$, regardless of the value of the constant I_1. The specific free energy f can be written in the following manner:

$$\begin{aligned} f = -T\Big\{\frac{3}{4}\ln 2 + \frac{1}{4\pi^2}\iint_0^{\pi} dp\,dx\,\ln[\cosh 2(K_2 + K_3)\sinh^2 2K_1 \\ - \sinh^2 2K_1 \cos x - 2\sinh 2(K_2 + K_3)\cosh 2k_1 \cos p \\ + 2(\cosh 2(K_2 + K_3) - \sinh 2K_2 \sinh 2K_3 \sin^2 p)]\Big\}. \end{aligned} \tag{4}$$

At $I_1 = I_2$ Eq. (4) can be reduced to the corresponding expression in Ref. 2. At $I_2 = I_3$ (4) becomes the free energy $f_I\{K_1, K_2\}$ of the rectangular Ising model. In the $I_3 = 0$ case the free energy can be expressed in terms of f_J:

$$f = -\frac{T}{4}\ln(4\,\mathrm{ch}\,2K_1) + \frac{1}{2}f_J\{K_1^*, K_2\},$$

where $\sinh 2K_1^* = \sinh^2 2K_1/2\cosh 2K_1$. AT $I_2 = -I_3$ the expression for f is symmetrical with respect to K_1 and K_2:

TABLE 1.

Relations between coupling constants	c	ϵ_0	s_0
$I_1, I_2 > -I_3$ $(T_c \neq 0)$	$8\left[\frac{I_1+I_3}{T}\right]^2 \exp\left\{-\frac{4(I_1+I_3)}{T}\right\}$	$-\frac{2I_1+I_2+I_3}{2}$	0
$I_2 > -I_3 > I_1$ $(T_c \neq 0)$	$\left[\frac{I_1+I_3}{T}\right]^2 \exp\left\{\frac{2(I_1+I_3)}{T}\right\}$	$-\frac{I_2-I_3}{2}$	0
$I_2 > -I_3 = I_1$ $(T_c \neq 0)$	$\frac{8}{5\sqrt{5}}\left[\frac{I_1-I_2}{T}\right]^2 \exp\left\{\frac{4(I_1-I_2)}{T}\right\}$	$-\frac{I_2-I_3}{2}$	$\frac{1}{4}\ln\frac{3+\sqrt{5}}{2}^*$
$I_2 = -I_3 > I_1$ $(T_c = 0)$	$\frac{4}{\pi}\left[\frac{I_1+I_3}{T}\right]^2 \exp\left\{\frac{2(I_1+I_3)}{T}\right\}$	$-I_2$	0
$I_1 > I_2 = -I_3$ $(T_c = 0)$	$\frac{4}{\pi}\left[\frac{I_1+I_3}{T}\right]^2 \exp\left\{-\frac{2(I_1+I_3)}{T}\right\}$	$-I_1$	0
$I_1 = I_2 = -I_3$ $(T_c = 0)$	$\frac{1}{\pi}\left[\frac{4I_1}{T}\right]^3 \exp\left\{-\frac{4I_1}{T}\right\}^*$	$-I_1$	$\frac{G^*}{\pi}$

$$f = -T\left\{\frac{3}{4}\ln 2 + \frac{1}{4\pi^2}\int_0^{\pi}\int dp\,dx\,\ln\left[\cosh^2 2K_1 + \cosh^2 2K_2 - \sinh^2 2K_1\cos x - \mathrm{sh}^2 2K_2\cos p\right]\right\}. \tag{5}$$

The symmetry in Eq. (5) is attributable to the fact that, as a result of the Mattis transformation, a lattice with constants I_1, I_2, and $-I_2$ can be converted to a lattice with constants I_2, I_1, and $-I_1$.

We represent the function $\phi(T) = -T^{-1}f$ [f is defined by Eq. (4)] in the following form:

$$\phi(T) = -T^{-1}\epsilon_0 + s_0 + g(T),$$

where ϵ_0 and s_0 are the specific energy and entropy at $T = 0$ and $g(T) \to 0$ as $T \to 0$. When $I_2 \neq I_3$ the low-temperature expansion of the $g(T)$ function is a series of exponential functions with exponents of the form $-\epsilon_i/T$. If, however, $I_2 = -I_3$, then the expansion of $g(T)$ has terms proportional to $1/T\exp\{-\epsilon_i/T\}$ in addition to the purely exponential terms. Table 1 gives the values associated with the first three terms in the low-temperature expansion of $\phi(T)$: the principal term of the asymptotic form of the heat capacity c (for the assumption $|I_1 - I_2|/I_1 \ll 1$, $|I_1 + I_3|/I_1 \ll 1$), ϵ_0, and s_0. (The formulas marked with an asterisk are contained in Ref. 2.) A simple combinatorial consideration shows that if $I_1 = -I_3 \neq I_1$ or $-I_3 > I_2$, then the total entropy S_0 of the lattice at $T=0$ is of the order of $\sqrt{N} \ln 2$ (N is the number of lattice sites), and $s_0 = 0$. We point out that the ground-state energy in the model with $-I_3 > I_1$ is the sum of the energies of one-dimensional Ising chains with coupling constants I_2 and I_3. At the same time, $I_1 \neq 0$ and $I_2 \neq -I_3$ such model has a phase transition.

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