Exact multisoliton solution of one-dimensional Landau-Lifshitz equations for an anisotropic ferromagnet

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The explicit form of multisoliton solutions of equations for the dynamics of an anisotropic ferromagnet is determined by using the Hirota method, with allowance for the magnetic-dipole interaction.

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Nonlinear dynamics of one-dimensional anisotropic ferromagnet can be described by the Landau-Lifshitz equation without dissipation[1]

$$\frac{\partial}{\partial t}\mathbf{M} = \left[\mathbf{M} \times \frac{\partial^2}{\partial x^2}\mathbf{M}\right] + \beta[\mathbf{M} \times \mathbf{n}_z](\mathbf{M}\mathbf{n}_z) - \gamma[\mathbf{M} \times \mathbf{n}_x](\mathbf{M}\mathbf{n}_x). \quad (1)$$

The gyromagnetic ratio, exchange interaction constant, and the nominal magnetization are introduced into the time renormalization t and the x coordinates, so that **M** is a dimensionless unit vector parallel to the magnetization direction and $\mathbf{n}_i$ are the unit vectors of the axes. The last terms on the right-hand side of Eq. (1) describe the anisotropy of the magnet and the magnetic-dipole interaction. In the special case of a single-axis ferromagnet with the anisótropy axis along the $\mathbf{n}_z$ ($\beta > 0$ corresponds to the easy-axis case) allowance for the magnetic-dipole interaction gives the value $\gamma = 4\pi$.

Lately, the solitary magnetization waves, described by the solutions of Eq. (1), have been widely discussed.[2-10] At $\beta > \gamma$ a simple solution of Eq. (1) such as $\mathbf{M} = \mathbf{M}(x - Vt)$ describes a moving domain boundary.[2] More complex, two-parameter, isolated solutions were examined in Ref. 3 for $\gamma = 0$ and in Refs. 4–6 for $\gamma = 4\pi$. These exact solutions are attributable to the fact that Eq. (1) is completely integrable. In fact, as shown in Ref. 7 for the case $\beta = \gamma = 0$, in Ref. 8 for $\gamma = 0$, and in Ref. 9 for the general case, the Landau-Lifshitz equation can be compared with the inverse problem of the scattering theory (ISP). Within the framework of the ISP, however, determination of the explicit form of the two-parameter solutions[3-6] is a complex computational problem. A more constructive method of obtaining exact, multisoliton solutions of nonlinear equations was proposed by Hirota and Satsuma.[10] In this work we obtain the Hirota transformation for the Landau-Lifshitz equation (1) and the explicit form of its N soliton solutions.

We shall seek a solution of Eq. (1) in the form

$$M_x + iM_y = 2\left\{\frac{g^*}{f} + \frac{f^*}{g}\right\}^{-1}, \qquad M_z = \left\{\frac{f^*}{g} - \frac{g^*}{f}\right\}\left\{\frac{g^*}{f} + \frac{f^*}{g}\right\}^{-1}, \tag{2}$$

where g and f are complex x and t functions. Substitution of Eq. (2) gives a rather cumbersome equation for g and f, but, as is usually the case in the Hirota method, it breaks up into a system of two simpler equations that have N soliton solutions. Using Hirota's differential operators,[10] we can write these equations in the following form:

$$f\left\{iD_t + D_x^2 - 1 - \frac{\epsilon}{2}\right\}gf^* + g^*\left\{D_x^2 gg - \frac{\epsilon}{2} f^*f^*\right\}, \tag{3a}$$

$$g^*\left\{-iD_t + D_x^2 - 1 - \frac{\epsilon}{2}\right\}gf^* + f\left\{D_x^2 f^*f^* - \frac{\epsilon}{2} gg\right\}, \tag{3b}$$

where

$$D_t^m D_x^n u(x,t)v(x,t) \equiv \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n u(x,t)v(x',t')\Big|_{\substack{x=x'\\ t=t'}}.$$

The anisotropy constant was introduced into the additional renormalization of x and t and the notation $\epsilon = \gamma/\beta$ was inserted. In contrast to the usual bilinear equations for the given method, expressions (3a) and (3b) are trilinear, which complicates further calculations but is not essential (see also Ref. 11).

Solution of Eqs. (3), which corresponds to the solitons, has the following form

$$g^* = \sum_{m=0}^{[(N-1)/2]} \sum_{{}_N C_{2m+1}} a(j_1,\ldots,j_{2m+1})\exp(\eta_{j_1} + \cdots + \eta_{j_{2m+1}}), \tag{4a}$$

$$f = \sum_{n=0}^{[N/2]} \sum_{{}_N C_{2n}} a(i_1, \ldots, i_{2n}) \exp(\eta_{i_1} + \ldots + \eta_{i_{2n}}), \tag{4b}$$

$$a(i_1, \ldots, i_n) = \begin{cases} \prod_{k<e}^{(n)} a(i_k, i_e) & n \geq 2 \\ 1 & n = 0, 1 \end{cases} \tag{5}$$

where $[N/2]$ is the maximum integer in addition to $N/2$, ${}_N C_n$ represents summation over all combinations of N elements in n, and (n) is the produce of all pair combinations of n elements. Each soliton corresponds to an exponential function with the exponent

$$\eta_i = k_i x + \omega_i t + \eta_i^{\circ}, \tag{6}$$

where

$$\omega_i^2 = (k_i^2 - 1)(1 + \epsilon - k_i^2). \tag{7}$$

For example, solution (4) for three solitons has the following form:

$$g^* = \exp\eta_1 + \exp\eta_2 + \exp\eta_3 + a(1, 2, 3)\exp(\eta_1 + \eta_2 + \eta_3),$$

$$f = 1 + a(1, 2)\exp(\eta_1 + \eta_2) + a(1, 3)\exp(\eta_1 + \eta_3) + a(2, 3)\exp(\eta_2 + \eta_3).$$

The factors $a(s, p)$, in terms of which the coefficients of the sums (4) are expressed, are

$$a(s, p) = \frac{k_p - k_s}{k_p + k_s} \; \frac{(\omega_s k_p^2 - \omega_p k_s^2) - (\omega_s - \omega_p)}{(\omega_s k_p^2 + \omega_p k_s^2) - (\omega_s + \omega_p)}. \tag{8}$$

The value $\ln a(s, p)$ determines the phase exchange between the s and p solitons passing through each other.

Solution of (4)–(8) describes the system of "free" solitons and solitons producing bound states in pairs. The free soliton in this case corresponds to the Bloch domain boundary, for which k_i and ω_i are real, $\mathrm{Re}\eta_i^0$ is an arbitrary value describing the location of the center of the domain wall and $\mathrm{Im}\eta_i^0$ is uniquely connected with k_i:

$$k_i^2 = 1 + \epsilon \cos^2 \mathrm{Im}\eta_i^{\circ}. \tag{9}$$

Thus, each soliton is characterized by one parameter (for example, by k_i) and the N-soliton solution is N parametric.

After substituting $k_i \to -k_i$ and $\omega_i \to -\omega_i$, we obtain a domain boundary that moves in the same direction with the same velocity, but in which $M_z \to -M_z$ (we call such solution an antisoliton).

Two domain boundaries of opposite sign (a soliton-antisoliton pair) can produce a bound state—self-localized magnetization wave. Thus, for this pair $k_i = k_j^*$, and the coupling between the phases η_i^0 and η_j^0 is described by the relation

$$k_i^2 = 1 + \epsilon \cosh^2\left\{\frac{1}{2}(\eta_i^0 - \eta_j^{0*})\right\}. \tag{10}$$

Of the four constants $\mathrm{Re}\eta_i^0$, $\mathrm{Re}\eta_j^0$, $\mathrm{Im}\eta_i^0$, and $\mathrm{Im}\eta_j^0$ two are arbitrary constants that determine the center-of-mass coordinate of the bound domain boundaries and the phase reference of their mutual oscillation.

We proved that solution of (4)–(10) satisfies the Landau-Lifshitz equation for the cases of one, two, and three solitons. Of course, the one-soliton solution coincides with Walker's results[2] for a moving or stationary domain boundary in a ferromagnet. The expression for the maximum velocity of motion of the wall $V_{\max} = (1+e)^{1/2} - 1$ in this case follows from Eqs. (6) and (7). The two- and three-soliton solutions with real k_i describe two- and three-domain walls, respectively, all of whose interaction reduces to the pair phase exchange occurring as a result of ther passage through each other (a special case of such two-soliton solution, in which the velocities of the two walls of opposite sign are the same in magnitude but opposite in direction, is given in Ref. 4. The expression for self-localized magnetization wave coincides with that obtained in Ref. 3 for $\gamma = 0$ and with the result of Ref. 6 for the Landau-Lifshitz equation with allowance for the magnetic-dipole interaction.

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