Long-wave asymptotic form of the many-body Green's functions of a one-dimensional Bose gas

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The continual integration method is used to obtain the long-wave asymptotic form of all the many-body Green's functions of a one-dimensional Bose gas with a point interaction.

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A one-dimensional Bose gas with point interaction is a completely integrable dynamic system. The corresponding nonlinear quantum Schrödinger equation can be solved by using the Bethe formulation[1] and the quantum inverse problem method.[2] Now the time has come to determine the Green's functions of completely integrable systems. The far this problem has been solved only for a system of impermeable bosons[3,4]—a Bose gas with an infinitely large interaction constant.

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In this paper we shall determine the asymptotic form of all the many-body Green's functions

$$<\psi(x_1, t_1) \ldots \psi(x_n, t_n)\, \bar{\psi}(x_1', t_1') \ldots \bar{\psi}(x_n', t_n')> \qquad (1)$$

for a Bose gas with an arbitrary interaction constant for $T=0$ in the limit $|x_i - x_j| \to\infty$. This method, previously developed[5,6] for describing two-dimensional and one-dimensional superfluid systems (see also Ref. 7), allows us to calculate the asymptotic form of the average (1) in the Euclidean region (with the substitution $t\to i\tau$). After defining

$$z_i = (x_i, \tau_i), \quad z_{n+i} = (x_i', \tau_i'), \quad i = 1, \ldots, n; \qquad (2)$$

$$|z_i - z_j|^2 = (x_i - x_j)^2 + c^2(\tau_i - \tau_j)^2, \qquad (3)$$

$$e_i = 1, \quad i = 1, \ldots, n; \; e_i = -1, \quad i = n+1, \ldots, 2n \qquad (4)$$

we can write the basic result for the Green's function (1) in the Euclidean region in the form

$$\prod_{i<j} (G(z_i - z_j))^{-e_i e_j} \qquad (5)$$

Here $G(z_i - z_j)$ is the one-particle Green's function which at large $|z_i - z_j|$ has the asymptotic form

$$G(z_i - z_j) \approx \rho \left| \frac{z_i - z_j}{R} \right|^{-\gamma}, \qquad (6)$$

$$\gamma = mc/2\pi\rho. \qquad (7)$$

In Eqs. (6) and (7) m is ths mass of the Bose particle, ρ is the density of the system, and c is the velocity of sound. The constant R was chosen in such a way that the coefficient in front of $|(z_i - z_j)/R|^{-\gamma}$ in (6) would be equal to ρ. The asymptotic form (6) for the one-particle function and the exponent (7) were obtained in Ref. 5. Equation (5) generalizes the asymptotic form (6) to the case of the n-body Green's function. We can interpret (5) as the distribution function of a system of $2n$ two-dimensional Coulomb charges $e_i = \pm 1$ that are located at points with the complex coordinates $z_j = x_i + ic\tau_i$ at a temperature γ.

For the derivation of Eq. (5) we write the average (1) as a continuous integral with respect to the field $\psi(x,\tau)$, $\bar{\psi}(x,\tau)$ and integrate with respect to fast and slow variables.[5,6] After integration with respect to the fast fields, we can write the average (1) in the Euclidean region in the form

$$<\psi_0(x_1, \tau_1) \ldots \psi_0(x_n, \tau_n)\, \bar{\psi}_0(x_1', \tau_1') \ldots \bar{\psi}_0(x_n', \tau_n')>_0. \qquad (8)$$

Here ψ_0 and $\bar{\psi}_0$ are the slow fields with Fourier components of less than some k_0 and $\langle ... \rangle_0$ represents an averaging over the slow fields with a weight $\exp S_h$, where S_h is the hydrodynamic action function that was calculated in Refs. 5 and 6.

In the integral with respect to the slow fields we switch to the phase-density variables

$$\psi_0(x,\tau) = \rho^{1/2}(x,\tau)e^{i\phi(x,\tau)}, \quad \bar{\psi}_0(x,\tau) = \rho^{1/2}(x,\tau)e^{-i\phi(x,\tau)}, \tag{9}$$

and we take S_h in the quadratic form[6]

$$\int dx\,d\tau \left(-\frac{p_\mu}{2m}(\partial_x\phi)^2 - \frac{p_{\mu\mu}}{2}(\partial_\tau\phi)^2 + ip_{\mu\rho_0}\pi\partial_\tau\phi + \frac{1}{2}p_{\rho_0\rho_0}\pi^2\right). \tag{10}$$

Here

$$\pi(x,\tau) = \rho(x,\tau) - \rho_0(k_0), \tag{11}$$

and $\rho_0(k_0)$ is determined by the condition $\partial p/\partial\rho_0 = 0$, where $p = S_h/\beta V$ is calculated for $\phi(x,\tau) = 0$, $\rho(x,\tau) = \rho_0 = \text{const}$. The coefficients p_μ, $p_{\mu\mu}$, $p_{\mu\rho_0}$, and $p_{\rho_0\rho_0}$ are the derivatives of p with respect to the chemical potential μ and the variable ρ_0. In the one-dimensional case for $T = 0$ the values $\rho_0(k_0)$, which is proportional to k_0^γ vanishes as $k_0 \to 0$. However, it is significant that $\rho_0(k_0)$ exists for all k_0 different from zero. This leads to the functional S_h (10) characteristic of superfluid Bose system.

We rewrite the average (8) in the variables ϕ and π

$$\left\langle \prod_{i=1}^{2n} [\rho_0(k_0) + \pi(z_i)]^{1/2} \exp i \sum_{i=1}^{2n} e_i\phi(z_i) \right\rangle_0 . \tag{12}$$

The values $\pi(z_i)$, which are small compared with $\rho_0(k_0)$, do not contribute to the first term of the asymptotic form, so that in a first approximation the first averaged factor in Eq. (12) is equal to $[\rho_0(k_0)]^n$. Taking S_h in the quadratic from (10), we obtain a Gaussian integral. As a result, for the average (12) we obtain

$$(\rho_0(k_0))^n \exp\left(-\frac{1}{2}\left\langle \left(\sum_i e_i\phi(z_i)\right)^2\right\rangle_0\right) =$$

$$(\rho_0(k_0))^n \exp\left\{-\frac{1}{2(2\pi)^2}\int_{|k|<k_0} dk\,d\omega\, g_{\phi\phi}(k,\omega)\left|\sum_i e_i e^{i(\omega\tau_i + kx_i)}\right|^2\right\}, \tag{13}$$

where

$$g_{\phi\phi}(k,\omega) = \frac{m}{\rho\left(k^2 + \dfrac{\omega^2}{c^2}\right)} \tag{14}$$

is the correlator of $\langle\phi\phi\rangle_0$ in the (k,ω) representation. We note that Eq. (14) of the $g_{\phi\phi}$ function is independent of the dimension of the system. After calculating the exponent in (13) in the limit $|z_i - z_j| \to \infty$, we obtain instead of (13)

$$(\rho_0(k_0))^n \exp\left\{ -\gamma n (\ln k_0 R + C) + \gamma \sum_{i<j} e_i e_j \ln (|z_i - z_j|/R) \right\}. \quad (15)$$

Here γ is the exponent of (7), C is the Euler constant, and R is the length constant [in fact Eq. (14) does not depend on thce choice of R]. Since (15) must be independent of the auxiliary parameter $k_0, \rho_0(k_0)$ must be proportional to k_0^γ. Let us now choose R in such a way that the following equation will be valid:

$$\rho_0(k_0) = \rho(k_0 R)^\gamma e^{\gamma C}. \quad (16)$$

Substitution of (16) in (13) gives the basic result (5).

The value of R can be calculated in the limit $\gamma \to 0$ corresponding to a Bose gas with a weak interaction (large density), which is opposite to the case of impermeable bosons, where the coupling constant is infinite and $\gamma = \frac{1}{2}$. In the limit $\gamma \to 0$.

$$R = (4mc)^{-1} e^{(2-C)} \approx 1{,}037\,(mc)^{-1}. \quad (17)$$

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