Production of sound waves in the early universe

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(Submitted March 28, 1980)
Pis'ma Zh. Eksp. Teor. Fiz. **31**, No. 11, 631-635 (5 June 1980)

A quantum theory of adiabatic perturbations of matter in the Friedman cosmology is formulated. It is shown that the mechanism of spontaneous phonon production near a cosmological singularity can lead to formation of the initial spectrum of adiabatic perturbations of matter.

PACS numbers: 98.80.Bp

The potential motion of an ideal fluid in the general theory of relativity can be described by a single scalar function -- the potential $\phi = \phi(x)$, whose gradient is proportional to the 4-momentum of a particle of matter:

$$wu_i = \phi_{,i}, \qquad (1)$$

where u^i is the 4-velocity of the matter ($u_i u^i = 1$),

$$w = \frac{\epsilon + p}{n} = (\phi_{,i}\,\phi_{,k}\,g^{ik})^{1/2} \qquad (2)$$

is the specific enthalpy, p is the pressure, ϵ is the density of matter;

$$n = \frac{dp}{dw} = \exp \int \frac{d\epsilon}{\epsilon + p} \qquad (3)$$

is the density of particle, $(nu^i)_{;i} = 0$ (the semicolon denotes the covariant derivative in the metric g_{ik}). All the quantities w, p, ϵ, and n are related to each other by the equation of state of the matter $p = p(w)$.

The Einstein equations $G_i^k = (\epsilon + p)u_i u^k - p\delta_i^k$, which describe the effect of the potential motion (1) on the space-time geometry, are obtained by equating to zero the first variation of the action ($c = \hbar = 8\pi G = 1$)

$$W = W[\phi, g^{ik}] = \int (p - \frac{1}{2} R)\sqrt{-g}\,d^4x \qquad (4)$$

in the metric g^{ik} for a specified function ϕ, $g = \det\{g_{ik}\}$, $R = R_i^i = -G_i^i$ is the scalar curvature.

Let us examine the small perturbations of the exact solutions of the type (1). In this case the ϕ function is the sum of the known $\phi^{(0)}$ function that defines the background solution, and the small function $\delta\phi = \Phi$ being analyzed:

$$\phi = \phi^{(o)} + \Phi, \quad g^{ik} = g^{ik(o)} - h^{ik}. \qquad (5)$$

The small tensor h^{ik} is a linear function of the scalar Φ. (No gravity waves are generated in first order with respect to Φ.) The Lagrangian of the linearized Einstein equations is obtained after expansion of the integrand (4) to second order in Φ and h^{ik}: where

$$W^{(2)} = W[\Phi, \psi_i^k] = \int L\sqrt{-g^{(o)}}\,d^4x,$$

$$L = \frac{\epsilon + p}{2}\dot{}(v_i v^i - 2v_i \psi_n^i u^k - \kappa^2(1-\beta^{-2}))$$

$$+\frac{\epsilon - p}{8}(\psi_{ik}\psi^{ik} - \frac{1}{2}\psi^2) + \frac{1}{8}(\psi_{ik;l}\psi^{ik;l} - 2\psi_{ik;l}\psi^{il;k} - \frac{1}{2}\psi_{,l}\psi^{,l}),$$

where

$$\psi_i^k = h_i^k - \tfrac{1}{2}h_l^l\delta_i^k, \quad \psi = \psi_i^i = -h_i^i, \quad v_i, = \Phi_{,i}/w,$$
$$\kappa = v_i u^i - \tfrac{1}{2}h_i^k u^i u_k, \quad \beta = \left(\frac{dp}{d\epsilon}\right)^{1/2} = \left(\frac{d\ln w}{d\ln n}\right)^{1/2} \tag{6}$$

is the velocity of sound. (All operations are done in the background metric g_{ik}; henceforth the superscript (0) is omitted.) The perturbed Einstein equations are obtained for the variation $\delta W^{(2)} = 0$ with respect to ψ_i^k for specified Φ and background metric:

$$\psi_{i;l}^{k;l} - \psi_i^{l;k}{}_l - \psi^{kl}{}_{;il} - \frac{1}{2}\psi_{;l}^{;l}\delta_i^k = (\epsilon - p)(\psi_i^k - \frac{1}{2}\psi\delta_i^k)$$
$$+ 2(\epsilon + p)[-v_i u^k - u_i v^k + \kappa(1-\beta^{-2})(u_i u^k - \frac{1}{2}\delta_i^k)]. \tag{7}$$

We are interested in the Cauchy problem of Eqs. (7). The background space is the homogeneous isotropic universe. Hydrodynamic perturbations in the Friedman models were first investigated by Lifshitz.[1] By using Lifshitz method, Field and Shepley[2] obtained a second-order equation for evolution of the Fourier component of the invariant density perturbations $\delta\epsilon$. Our aim is to construct two, canonically conjugate invariant scalars which describe the evolution of the two physical degrees of freedom of the freedom of the potential perturbations and which are analogs of the velocity potential and density perturbations in a steady-state (Newtonian) medium.[3]

We choose a synchronous frame of reference in which the $\phi^{(0)}$ scalar depends on the universal time t:

$$ds^2 = dt^2 - a^2(\delta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta \tag{8}$$

$\delta_{\alpha\beta}$ is a unit tensor, and the function $a = a(t)$ is defined by the equalities

$$\epsilon = 3\left(\frac{\dot{a}}{a}\right)^2, \quad \epsilon + p = -2\left(\frac{\dot{a}}{a}\right)^{\cdot}, \tag{9}$$

where $(\cdot) = \partial/\partial t \cdot h_{\alpha\beta}$ is a small tensor in the Euclidean space $\mathbf{x} = \{x^\alpha\} = (x,y,z)$. The general form of $h_{\alpha\beta}$ (A and B are scalars that are linear in Φ) is

$$h_{\alpha\beta} = A\,\delta_{\alpha\beta} + B_{,\alpha\beta}. \tag{10}$$

For the hydrodynamic quantities we have

$$\frac{\delta w}{w} = \frac{\Phi}{w} = \dot{v} - 3\beta^2 \frac{a}{a} v , \quad \delta u^{\circ}{}_{\bullet} = 0, \quad u_{\alpha} = v_{,\alpha} , \tag{11}$$

where $v = \Phi / w$. The gauge freedom in selecting the scalars v, A, and B, which is due to arbitrariness in constructing the synchronous frame of reference (8), has the form[4]:

$$\tilde{v} = v + \frac{1}{2} F , \quad \tilde{A} = A + \frac{\dot{a}}{a} F , \quad \tilde{B} = B + F \int \frac{dt}{a^2} , \tag{12}$$

where $F = F(\mathbf{x})$ is an arbitrary, small function of the spatial coordinates. It follows from this that the scalar

$$q = q(t, \mathrm{x}) = 3\left(\frac{\dot{a}}{a} v - \frac{1}{2} A \right) \tag{13}$$

is gauge invariant. We obtain from Eqs. (7) the formulas relating the scalars v, A, and B to the q function:

$$v = \frac{a}{3a} q - \frac{1}{2} Q , \quad A = -\frac{a}{a} Q ,$$

$$B = -\frac{1}{a^2} Q + \frac{1}{a^3} \int a \gamma q \, dt , \quad Q = Q(t, \mathrm{x}) = \int \gamma q \, dt \tag{14}$$

and the equation for the q scalar, which describes the evolution of the two physical degrees of freedom of the potential perturbations of an ideal fluid in the Friedman spatially plane cosmology is

$$q'' + 2 \frac{\xi'}{\xi} q' - \beta^2 \Delta q = 0, \tag{15}$$

where

$$\gamma = 1 + \frac{p}{\epsilon}, \; \xi = \frac{a}{\beta}\left(\frac{\gamma}{3}\right)^{1/2}, \; (') = a\frac{\partial}{\partial t}, \; \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

After substituting Eq. (14) in Eq. (6), rather long calculations lead us to conclude that the Lagrangian

$$\tilde{L} = \tilde{L}(q) = \frac{\xi^2}{2a^4} \left(q'^2 - \beta^2 q_{,\alpha} q^{,\alpha} \right), \tag{16}$$

whose variation with respect to q ($\delta W^{(2)} = 0$) leads to Eq. (15), differs from L only in the divergence term. We denote by $\sigma = \sigma(t,\mathbf{x}) = \partial a^3 \tilde{L} / \partial \dot{q} = \alpha^2 q'$ the gauge invariant scalar that is canonically conjugated with q. We have from Eq. (11)

$$\frac{\delta \epsilon}{\epsilon} = \gamma \frac{\delta n}{n} = \frac{\sigma}{a a^2} + \gamma \left(-q + \frac{3}{2} \frac{a}{a} Q \right) \tag{17}$$

The canonical quantization of the potential perturbations (sound waves), which is based on the Lagrangian $\tilde{L}$, predicated on the simultaneous commutation relation for the canonically conjugated operators q and σ:

$$[q(t,\mathbf{x})\sigma(t,\mathbf{y})] = q\sigma - \sigma q = i\,\delta(\mathbf{x}-\mathbf{y}), \tag{18}$$

which is analogous to the commutation rule between the operators of the velocity potential (14) and the perturbed density (17) in a stationary medium.[3]

In the representation of the occupation numbers of phonons with a specified 3-momentum $\mathbf{k}$, the field operator q has the form ($k = |\mathbf{k}|$)

$$q = (2\pi)^{-3/2} \int d^3\mathbf{k}\,(a_\mathbf{k}\, e^{i\mathbf{kx}}\, \nu_\mathbf{k}(t) + a_\mathbf{k}^+\, e^{-i\mathbf{kx}}\, \nu_k^*(t)), \tag{19}$$

where $a_\mathbf{k}$ and $a_\mathbf{k}^+$ are the phonon annihilation and creation operators that satisfy the Bose commutation relations [see Eq. (18)]

$$[a_\mathbf{k} a_{\mathbf{k}'}] = 0, \quad [a_\mathbf{k} a_{\mathbf{k}'}^+] = \delta(\mathbf{k}-\mathbf{k}'). \tag{20}$$

An important physical conclusion follows from Eq. (15): the equation for sound waves in an expanding universe is conformally invariant to the equation for sound propagation in a flat world only when $\alpha'' = 0$. A universe filled by ultrarelativistic particles with the equation of state $p = \epsilon/3$ has such a property. No phonons are produced in this world. When $\alpha'' \neq 0$, a spontaneous production of sound oscillations occurs during cosmological expansion. In principle, this mechanism can lead to formation of the initial spectrum of density perturbations within the context of the adiabatic theory of formation of galaxies[5] as a result of generation of long waves. We note that the outward similarity of Eq. (15) to the corresponding equation for the amplitude of the gravity waves in the Friedman model makes it possible to transfer to the phonon case the basic conclusions for the long-wave part of the spectrum.[6,7]

We obtain from Eqs. (17)–(20) the correlation function for the density fluctuation in the cross section $t = \text{const}$ in the asymptotic region $t \gg 1$ [in the ordinary units $t \gg (G\hbar/c^5)^{1/2}$ in the expansion stage $p = \epsilon/3$ ($a = \sqrt{t}$)]:

$$< \frac{\delta\epsilon}{\epsilon}(\mathbf{x})\, \frac{\delta\epsilon}{\epsilon}(\mathbf{x}') > = C_0 \int_{k_D}^{\infty} k^3\, dk\, |\beta_k|^2\, \frac{\sin\zeta}{\zeta}, \tag{21}$$

where $c_0 = (1/\pi^2)(4/3)^{3/2}$, $\zeta = k\,|\mathbf{x}-\mathbf{x}'|$, and β_k is a nonadiabatic amplification factor calculated in the classical problem of the scattering of a wave αq [see Eq. (15)] by the effective potential $U = \xi''/\xi$. The integration in Eq. (21) is extended to scales than the Dzhinskii scale $k_D = (3/4t)^{1/2}$.

The author thanks Ya. B. Zel'dovich, I. D. Novikov, and D. A. Kompaneits for useful discussions of the investigation and valuable comments.

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